Input	Output
0	7
1	11
2	15
3	19
4	23

Input	Output	Δ
0	7	4
1	11	4
2	15	4
3	19	4
4	23	

- *How do you help your students see that "constant difference" implies "linear fit"?*

- *Does "linear fit" imply "constant difference"?*

Input	Output	Δ
0	7	4
1	11	4
2	15	4
3	19	4
4	23	

Note: the rule

$$x \mapsto 7 + 28\,x - 50\,x^2 + 35\,x^3 - 10\,x^4 + x^5$$

also matches the above table. More about this later.

What if there isn't a constant difference?
Then you have to use your wits.

Input	Output	Δ
0	3	7
1	10	11
2	21	15
3	36	19
4	55	

(This is the problem on page 2.)

Input	Output	Δ	Δ^2
0	3	7	4
1	10	11	4
2	21	15	4
3	36	19	
4	55		

Input	Output	Δ	Δ^2
0	3	7	4
1	10	11	4
2	21	15	4
3	36	19	
4	55		

- How did you find a rule that agrees with this table?
- Do constant second differences imply quadratic fit?
- Does a quadratic fit imply constant second differences?
- Is there only one rule that agrees with this table?

Try some on page 3 . . .

Note:

$$n \mapsto 3 + 29\,n - 48\,n^2 + 35\,n^3 - 10\,n^4 + n^5$$

also agrees with the table. More about that later . . .

Input	Output	Δ	Δ^2
0	3	7	4
1	10	11	4
2	21	15	4
3	36	19	
4	55		

Suppose someone walks up to you and says "How about this one?"

n	$f(n)$	Δ	Δ^2	Δ^3
0	1	-2	14	12
1	-1	12	26	12
2	11	38	38	12
3	49	76	50	12
4	125	126	62	12
5	251	188	74	
6	439	262		
7	701			

Now what? Let's work on this as a group.

n	$f(n)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6	Δ^7
0	1	-2	14	12	0	0	0	0
1	-1	12	26	12	0	0	0	
2	11	38	38	12	0	0		
3	49	76	50	12	0			
4	125	126	62	12				
5	251	188	74					
6	439	262						
7	701							

n	$f(n)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6	Δ^7
0	1	-2	14	12	0	0	0	0
1	-1	12	26	12	0	0	0	
2	11	38	38	12	0	0		
3	49	76	50	12	0			
4	125	126	62	12				
5	251	188	74					
6	439	262						
7	701							

$$f(4) = \mathbf{1} \cdot 1 + \mathbf{4} \cdot (-2) + \mathbf{6} \cdot 14 + \mathbf{4} \cdot 12 + \mathbf{1} \cdot 0$$

$$f(5) = \mathbf{1} \cdot 1 + \mathbf{5} \cdot (-2) + \mathbf{10} \cdot 14 + \mathbf{10} \cdot 12 + \mathbf{5} \cdot 0 + \mathbf{1} \cdot 0$$

$$f(6) = \mathbf{1} \cdot 1 + \mathbf{6} \cdot (-2) + \mathbf{15} \cdot 14 + \mathbf{20} \cdot 12 + \mathbf{15} \cdot 0 + \mathbf{6} \cdot 0 + \mathbf{1} \cdot 0$$

Brief Review of Pascal's Triangle

- *Pascal's triangle is a recursively generated number pattern.*

$$\binom{0}{0} = 1$$

$$\binom{1}{0} = 1 \quad \binom{1}{1} = 1$$

$$\binom{2}{0} = 1 \quad \binom{2}{1} = 2 \quad \binom{2}{2} = 1$$

$$\binom{3}{0} = 1 \quad \binom{3}{1} = 3 \quad \binom{3}{2} = 3 \quad \binom{3}{3} = 1$$

$$\binom{4}{0} = 1 \quad \binom{4}{1} = 4 \quad \binom{4}{2} = 6 \quad \binom{4}{3} = 4 \quad \binom{4}{4} = 1$$

$$\binom{5}{0} = 1 \quad \binom{5}{1} = 5 \quad \binom{5}{2} = 10 \quad \binom{5}{3} = 10 \quad \binom{5}{4} = 5 \quad \binom{5}{5} = 1$$

Each row starts and ends with 1, and any interior element is the sum of the two above it.

In symbols:

$$\binom{n}{k} = \begin{cases} 1 & \text{if } k = 0 \text{ or if } k = n \\ \binom{n-1}{k-1} + \binom{n-1}{k} & \text{if } 0 < k < n \end{cases}$$

Question: What is $\binom{7}{3}$?

- *The entries in Pascal's triangle are quotients of factorials.* There's an explicit formula for $\binom{n}{k}$ in terms of factorials:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\text{So, } \binom{12}{5} = \frac{12!}{5!\,7!}$$

$$= \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \ \times \ 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$= \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$= 792$$

Question: What is $\binom{12}{7}$?

- *The entries in Pascal's triangle are rational expressions.* Sometimes, it's useful to do the cancellations in the factorial expression above and write $\binom{n}{k}$ as a product of factors:

$$\binom{n}{k} = \frac{n(n-1)(n-2)(n-3)\ldots(n-k+1)}{k!}$$

One advantage of this expression over all the others is that n can be any real number here. In a sense, this expression extends the formula for entries in Pascal's triangle from integers to real (or even complex) numbers, so we can write $\binom{x}{k}$:

$$\binom{x}{3} = \frac{x(x-1)(x-2)}{3!} = \frac{2\,x - 3\,x^2 + x^3}{6}$$

k	$\binom{x}{k}$	Expanded version
0	1	1
1	x	x
2	$\dfrac{x(x-1)}{2!}$	$\dfrac{-x+x^2}{2}$
3	$\dfrac{x(x-1)(x-2)}{3!}$	$\dfrac{2\,x-3\,x^2+x^3}{6}$
4	$\dfrac{x(x-1)(x-2)(x-3)}{4!}$	$\dfrac{-6\,x+11\,x^2-6\,x^3+x^4}{24}$
5	$\dfrac{x(x-1)(x-2)(x-3)(x-4)}{5!}$	$\dfrac{24\,x-50\,x^2+35\,x^3-10\,x^4+x^5}{120}$
6	$\dfrac{x(x-1)(x-2)(x-3)(x-4)(x-5)}{6!}$	$\dfrac{-120\,x+274\,x^2-225\,x^3+85\,x^4-15\,x^5+x^6}{720}$

(On the TI-89, type `nCr(x,3)`.)

n	$f(n)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6	Δ^7
0	1	−2	14	12	0	0	0	0
1	−1	12	26	12	0	0	0	
2	11	38	38	12	0	0		
3	49	76	50	12	0			
4	125	126	62	12				
5	251	188	74					
6	439	262						
7	701							

$$f(4) = \binom{4}{0} \cdot 1 + \binom{4}{1} \cdot (-2) + \binom{4}{2} \cdot 14 + \binom{4}{3} \cdot 12 + \binom{4}{4} \cdot 0$$

$$f(5) = \binom{5}{0} \cdot 1 + \binom{5}{1} \cdot (-2) + \binom{5}{2} \cdot 14 + \binom{5}{3} \cdot 12 + \binom{5}{4} \cdot 0 + \binom{5}{5} \cdot 0$$

$$\vdots$$

$$f(n) = \binom{n}{0} \cdot 1 + \binom{n}{1} \cdot (-2) + \binom{n}{2} \cdot 14 + \binom{n}{3} \cdot 12 + \binom{n}{4} \cdot 0 + \binom{n}{5} \cdot 0 + \dots$$

So, we have a conjectured “fit”

$$f(x) = \binom{x}{0} \cdot 1 + \binom{x}{1} \cdot (-2) + \binom{x}{2} \cdot 14 + \binom{x}{3} \cdot 12$$

$$= 1 \cdot 1 + x \cdot (-2) + \frac{x(x-1)}{2} \cdot 14 + \frac{x(x-1)(x-2)}{6} \cdot 12$$

On the TI-89,

```
nCr (x, 0)*1 +  nCr (x, 1)*(-2) +
                    nCr (x, 2)*14  +  nCr (x, 3)*12
```

produces

$$1 - 5\,x + x^2 + 2\,x^3$$

… and it works.

Example

Input	Output	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0	1	5	52	192	264	120
1	6	57	244	456	384	120
2	63	301	700	840	504	120
3	364	1001	1540	1344	624	
4	1365	2541	2884	1968		
5	3906	5425	4852			
6	9331	10277				
7	19608					

The "0th" row is $\{1, 5, 52, 192, 264, 120\}$, so, with no further fuss, here's a (degree 5) polynomial function that agrees with the table:

The "0th" row is $\{1, 5, 52, 192, 264, 120\}$

$$f(x) = 1 \cdot \binom{x}{0} + 5 \cdot \binom{x}{1} + 52 \cdot \binom{x}{2} + 192 \cdot \binom{x}{3} + 264 \cdot \binom{x}{4} + 120 \cdot \binom{x}{5}$$

$$= 1 + 5x + 52\frac{x(x-1)}{2} + 192\frac{x(x-1)(x-3)}{6} + 264\frac{x(x-1)(x-2)(x-3)}{24}$$

$$+ 120\frac{x(x-1)(x-2)(x-3)(x-4)}{120}$$

...with a little help from a friend

$= 1 + x + x^2 + x^3 + x^4 + x^5$ (and *this* works)

Many people call the call the polynomials $\binom{x}{k}$ the "combinatorial polynomials" or the "Mahler polynomials."

This has nothing to do with the composer Gustav Mahler; rather it is in honor of the German-born mathematician Kurt Mahler (1903–1988). In the 1960s, Mahler showed that these polynomials could be used to extend some important classical functions to a number system useful in number theory (known as the "p-adic numbers").

This method is sometimes known as *Newton's Difference Formula*, because it was made famous by, well, Isaac Newton (although it is probably older than that).

In 1860, George Boole (of Boolean logic fame) published what may have been the first discrete mathematics text:

A Treatise on the Calculus of Finite Differences.

In it, Boole derived Newton's Difference Formula and a whole lot more.

Try some: Pick a couple problems on page 9 of the handout (and do them).

One More Thing

Input	Output
1	-6
3	166
6	7159
8	31291
9	56938
12	243787

$$
\begin{aligned}
f(x) = \; & A(x-1)(x-3)(x-6)(x-8)(x-9) \\
& + B(x-1)(x-3)(x-6)(x-8)(x-12) \\
& + C(x-1)(x-3)(x-6)(x-9)(x-12) \\
& + D(x-1)(x-3)(x-8)(x-9)(x-12) \\
& + E(x-1)(x-6)(x-8)(x-9)(x-12) \\
& + F(x-3)(x-6)(x-8)(x-9)(x-12) \quad (*)
\end{aligned}
$$

The numbers A–F will be determined in a minute. Each product is formed by taking

$$(x-1)(x-3)(x-6)(x-8)(x-9)(x-12)$$

and "dropping" one factor. So, each product is a polynomial of degree 5, and *that* ensures that the whole expression will be of degree at most 5.

Why write f is such a messy way? Well, it allows you to easily calculate $f(n)$ for any n between 0 and 5. For example, from the table, we want $f(1)$ to be -6. So, we calculate like this:

$$\begin{aligned} f(1) = -6 = \;& A(1-1)(1-3)(1-6)(1-8)(1-9) \\ &+ B(1-1)(1-3)(1-6)(1-8)(1-12) \\ &+ C(1-1)(1-3)(1-6)(1-9)(1-12) \\ &+ D(1-1)(1-3)(1-8)(1-9)(1-12) \\ &+ E(1-1)(1-6)(1-8)(1-9)(1-12) \\ &+ F(1-3)(1-6)(1-8)(1-9)(1-12) \end{aligned}$$

But look: all the terms except the last have a factor of 0, so they all vanish. We get

$$-6 = F(-2)(-5)(-7)(-8)(-11) = -6160F$$

So,

$$F = \frac{-6}{-6160} = \frac{3}{3080} \ldots$$

$$
\begin{aligned}
f(x) = \ & \frac{243787}{7128}(x-1)(x-3)(x-6)(x-8)(x-9) \\
& - \frac{28469}{216}(x-1)(x-3)(x-6)(x-8)(x-12) \\
& + \frac{31291}{280}(x-1)(x-3)(x-6)(x-9)(x-12) \\
& - \frac{7159}{540}(x-1)(x-3)(x-8)(x-9)(x-12) \\
& + \frac{83}{810}(x-1)(x-6)(x-8)(x-9)(x-12) \\
& + \frac{3}{3080}(x-3)(x-6)(x-8)(x-9)(x-12)
\end{aligned}
$$

A CAS simplifies this to $x^5 - 3x^3 + x^2 - 5$. Try it for yourself.

A radio show offered a prize to the first caller who could predict the next term in this sequence:

$$\{1, 2, 4, 8, 16\}$$

(1) What would you get if you used "common sense?"

(2) What would you get if you used Newton's difference formula or Lagrange interpolation?

Find a function that agrees with this table:

Input	Output
0	7
1	11
2	15
3	19
4	23
5	267

Compare your answer with problem 1 on page 1.

The local radio station AM 850 was giving away a Binky Baby prize for the first listener who could call in and answer the following question:

> What's the next term in the sequence

$$1, 4, 8, 9, 27 \ldots ?$$

The answer the radio station was looking for?

42

1931: The Red Sox players first wear numbers on their uniforms. Since then, the Red Sox have retired five uniform numbers:

- Ted Williams' # 9 and Joe Cronin's # 4 officially retired May 29, 1984;
- Bobby Doerr's # 1 retired May 21, 1988;
- Carl Yastrzemski's # 8 retired August 6, 1989; and
- Carlton Fisk's # 27 retired September 4, 2000.

Major League Baseball retired the # 42 of Jackie Robinson (in April of 1997 to commemorate the 50th anniversary of the integration of Major League Baseball). The numbers have been placed in increasing order above right field.

The Moral of the Story

There is ***no single right answer*** *to the problem "what's the next term in the sequence?" The methods we developed today show that there are infinitely many polynomial functions that agree with any table—there are infinitely many ways to "continue a pattern."*

8 Computers are designed around off/on switches that are used to represent numbers. In the following pattern, which represents the numbers from 0 to 10, ○ represents a switch that is on and ● represents a switch that is off.

● ● ● ● =0
● ● ● ○ =1
● ● ○ ● =2
● ● ○ ○ =3
● ○ ● ● =4
● ○ ● ○ =5
● ○ ○ ● =6
● ○ ○ ○ =7
○ ● ● ● =8
○ ● ● ○ =9
○ ● ○ ● =10

Which of the following represents the number 11?

A. ○ ○ ○ ●

B. ○ ○ ● ●

C. ○ ● ○ ○

D. ○ ○ ○ ○

Reporting Category for Item 8: ***Patterns, Relations, and Algebra***

Note: The *Mathematics Teacher* has a new department called *Delving Deeper*. The NCTM blurb says:

> *This department focuses on mathematics content relevant to secondary school teachers. It provides a forum that allows classroom teachers to share their mathematics from their work with students, classroom investigations and projects, and other experiences. We encourage submissions that pose and solve a novel or interesting mathematics problem, expand on connections among different mathematical topics, present a general method for describing a mathematical notion or a class of problems, elaborate on new insights into familiar secondary school mathematics, or leave the reader with a mathematical idea to expand. Please send submissions to*

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