Focus on Mathematics Seminar
Waltham High School
February 4, 2008

Close Enough: Solving Equations Iteratively

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Using the solver that is built into a graphing calculator:

Let's begin with with a short introduction to the equation solver that is built into most graphing calculators by solving the (famous) equation

$$x^3 - 2x - 5 = 0.$$

The equation solver can "solve" an equation with one or more variables given values for all of the variables except one. It is located on the **MATH** menu. When you select **0:Solver**, you will see one of two screens. If the **eqn** variable is empty, you will see the equation editor. If an equation is already stored in **eqn**, you will see the interactive solver editor.

The **zero** function on the **CALC** menu is a graphical interface to the solver.

The solver can also be used directly on the home screen.

Page two of this document was page 1 of the Algebra I activity (#14) "Pythagorean Theorem with Equation Solver" (May 7, 2007). It is available at

`http://tialgebra.com/activitiesarchive/`

Page three of this document was page 2 of the Algebra I activity (#14) “Pythagorean Theorem with Equation Solver” (May 7, 2007). It is available at

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How does the solver work?

The solver probably is an implementation of a numerical algorithm called Brent's method. Brent's method was developed in 1973 and was based on an algorithm developed by van Wijngaarden, Dekker, and others at the Mathematical Center in Amsterdam a few years earlier. It is a somewhat elaborate combination of three different numerical methods, and we will work with two of them today.

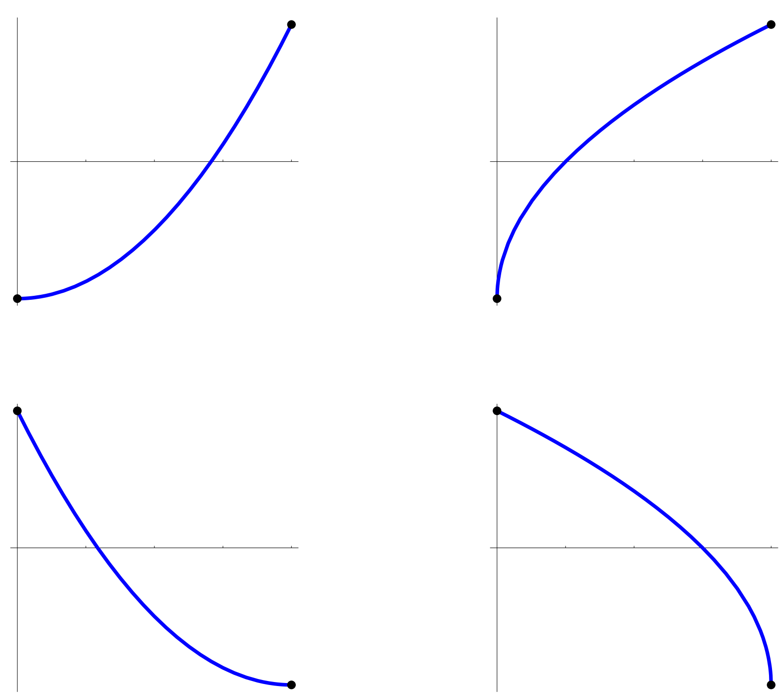
The method of bisection

This method is reminiscent of the following "modified" MCAS question (similar to 2004 Question 14):

Which of the following is closest to $\sqrt{56}$?

(a) 6.7 (b) 7.3 (c) 7.7 (d) 8.3

In the method of bisection, we solve the equation $f(x) = 0$ by "bracketing" a root and repeatedly subdividing until we find a sufficiently accurate approximation to the root. In other words, we start with an interval $a \leq x \leq b$ such that $f(a) < 0$ and $f(b) > 0$ or vice versa. In either case, $f(a)f(b) < 0$. Then we calculate the value of $f(x)$ at the midpoint m of the interval. We replace either a or b with m so that the function values still have opposite signs at the endpoints of the new interval.



Example 1. Let's start with $f(x) = x^3 - 2x - 5 = 0$. Then $f(0) = -5$ and $f(3) = 16$. To get ready to use bisection, set **Y₁=X^3-2X-5**, store **0→A** and **3→B**. In this example, let's approximate the root to four decimal places, so set **FLOAT** to 4 in the **MODE** menu.

We'll do this first example "by hand," and to save us some keystrokes, we'll take advantage of the fact that we can enter more than one command on one line:

(A+B)/2→M:Y₁(M)

We get the value of $f(x)$ at the midpoint **M** of **A** and **B**. In this case, that value is negative, so we replace **A** by **M** by executing

M→A

Now we take advantage of the fact that we can reexecute previous commands by hitting **2nd ENTRY** twice. We get

(A+B)/2→M:Y₁(M)

back again and now we hit **ENTER**. We get the value of $f(x)$ at the (new) midpoint **M** of the new **A** and **B**. Now this value is positive, so we replace **B** by the new **M**. Again hitting **2nd ENTRY** twice saves some keystrokes.

k	a_k $f(a_k) < 0$	b_k $f(b_k) > 0$	sign$(f(m_k))$ (+ or −)
0	0	3	
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			
13			
14			
15			
16			
17			
18			
19			
20			

More examples:

As you try the following examples, you may want to use a relatively short (but not completely general) program. The version given below assumes that $f(a_k) < 0$ and $f(b_k) > 0$ (as in the top two figures on page 4). How should it be changed if $f(a_k) > 0$ and $f(b_k) < 0$?

```
:(A+B)/2→M
:Y1(M)→F
:Disp F
:If F<0
:Then
:M→A
:Else
:M→B
:End
:Disp {A,B}
```

In these examples, you may want to stop after finding the roots to two decimal places, especially if you don't want to use the program given above.

Example 2. Solve $x^3 + 3x^2 - 1 = 0$. (How many solutions are there?)

k	a_k sign$(f(a_k))$?	b_k sign$(f(b_k))$?	sign$(f(m_k))$ (+ or −)
0			
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			
13			
14			
15			

Example 2 continued. $x^3 + 3x^2 - 1 = 0$

k	a_k sign$(f(a_k))$?	b_k sign$(f(b_k))$?	sign$(f(m_k))$ (+ or $-$)
0			
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			
13			
14			
15			

k	a_k sign$(f(a_k))$?	b_k sign$(f(b_k))$?	sign$(f(m_k))$ (+ or $-$)
0			
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			
13			
14			
15			

Example 3. $27x^3 + 54x^2 - 32 = 0$ (How many solutions are there?)

k	a_k sign$(f(a_k))$?	b_k sign$(f(b_k))$?	sign$(f(m_k))$ (+ or $-$)
0			
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			

Example 4. $\cos x = x$

k	a_k sign$(f(a_k))$?	b_k sign$(f(b_k))$?	sign$(f(m_k))$ (+ or $-$)
0			
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			

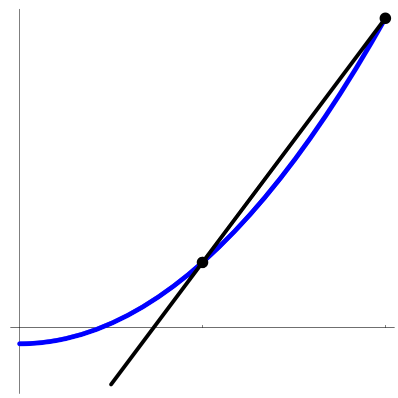
Example 5. $\tan x = x$ (Find one solution near π and one near 2π.)

k	a_k sign$(f(a_k))$?	b_k sign$(f(b_k))$?	sign$(f(m_k))$ (+ or $-$)
0			
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			

k	a_k sign$(f(a_k))$?	b_k sign$(f(b_k))$?	sign$(f(m_k))$ (+ or $-$)
0			
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			

The secant method

The secant method uses the two previous iterates to produce the next iterate. Given two points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ on the graph, we form the line through those two points and the next iterate x_2 is where this line crosses the x-axis.



Derive the formula for x_2 in terms of the four numbers x_0, $f(x_0)$, x_1, and $f(x_1)$.

In general, the secant method is given by

$$x_{k+1} = x_k - f(x_k)\left[\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}\right].$$

To do this method "by hand" on the calculator, start by setting $\mathbf{Y_1}$ equal to $f(x)$, store x_0 in **A**, x_1 in **B**, and $f(x_0)$ in **C**.

Now enter the following commands (all on one line):

Y₁(B)→D:(B-A)/(D-C)→N:B→A:D→C:A-N∗D→B: {B,Y₁(B)}

Let's redo Example 1 using the secant rule.

Example 1. $x^3 - 2x - 5 = 0$

k	x_k	$f(x_k)$
0	0	-5
1	3	16
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		
13		
14		
15		

Example 2. $x^3 + 3x^2 - 1 = 0$

k	x_k	$f(x_k)$
0		
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		
13		
14		
15		

k	x_k	$f(x_k)$
0		
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		
13		
14		
15		

Example 2 continued. $x^3 + 3x^2 - 1 = 0$

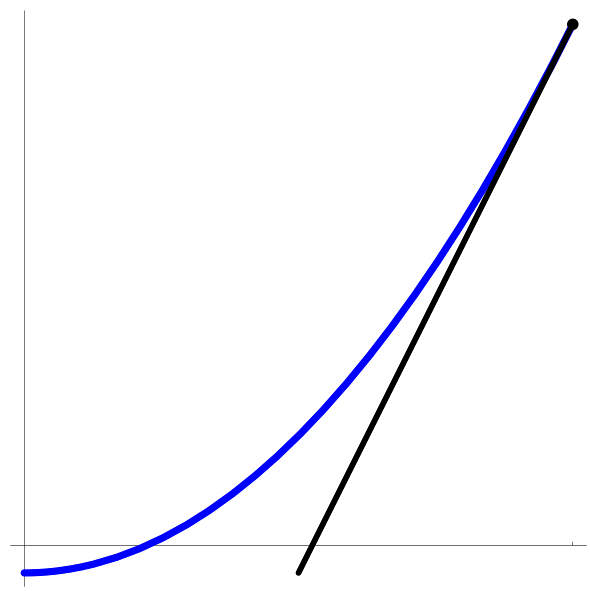
k	x_k	$f(x_k)$
0		
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		
13		
14		
15		

Example 3. $27x^3 + 54x^2 - 32 = 0$

k	x_k	$f(x_k)$
0		
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		
13		
14		
15		

Newton's method

The secant method requires two points on the graph. Newton's method uses lines tangent to the graph rather than secant lines. One way to think about Newton's method is to start with the secant method and let the two points on the graph come together along the graph. The secant line approaches a tangent line.



Starting with the formula

$$x_{k+1} = x_k - f(x_k)\left[\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}\right]$$

for the secant method, we note that the fraction inside the brackets is the inverse of the slope of the secant line. If we let x_{k-1} approach x_k, then that slope approaches the inverse of the slope of the tangent line to the graph of $f(x)$ at x_k. In calculus, the slope of the tangent line at x_k is denoted $f'(x_k)$, so Newton's method is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

Fact: If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$, then

$$f'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + a_1.$$

We can do Newton's method on the calculator without any preliminary steps. For example, suppose that we want to solve $f(x) = x^3 - 2x - 5 = 0$ again.

Store your initial guess for the root in **X** and then enter

X-(X^3-2X-5)/(3X^2-2)→X

or if you also want to see the function values, you can enter

X-(X^3-2X-5)/(3X^2-2)→X:{X,X^3-2X-5}

Pressing **ENTER** repeatedly iterates Newton's method.

Back to Example 1 once again.

Example 1. $x^3 - 2x - 5 = 0$

k	x_k	$f(x_k)$
0	0	-5
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		
13		
14		
15		

Example 6. Newton's method has trouble with $x^3 - 2x + 2 = 0$. Note what happens if $x_0 = 0$.

Back to Example 1. How many solutions does $x^3 - 2x - 5 = 0$ have?

k	x_k	$f(x_k)$
0		
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
11		
12		
13		
14		
15		
16		
17		
18		
19		
20		

So how does the solver work?

Brent's method is a combination of bisection and the secant method. It does not use Newton's method because it does not know the slope of the tangent lines for arbitrary functions. At each step, the method uses both bisection and the secant method. If the secant method gives a better result than bisection, then that result is used. Otherwise, the result of bisection is used. (Actually a third, slightly faster method similar to the secant method is also used.)