NCTM EASTERN REGIONAL CONFERENCE
HARTFORD, CT OCTOBER 7, 2005

The Geometry of Voting

Mike Spillane
Watertown High School

Dan Wulf
Watertown High School

Steve Rosenberg
Boston University

2003-2004 **FOCUS ON MATH** Study Group
www.focusonmath.org/FOM/programs/seminars

FOCUS ON MATH is funded by
the National Science Foundation under grant no. 0314692

The Geometry of Voting

Mike Spillane (Watertown HS)
Dan Wulf (Watertown HS)
Steve Rosenberg (BU)
Focus on Math: www.focusonmath.org/FOM/programs/seminars

Reference: *Geometry, Voting, and Paradoxes*; Saari, Donald G. and Valongnes, Fabrice, Mathematics Magazine 71 (1998), 243-259.

Plurality Method
<u>(most votes wins)</u>

Please vote for your favorite of the three listed choices.

 Soda ☐  Beer ☐  Wine ☐

Next please fill in the totals for everyone participating today.

Totals

Soda	Beer	Wine

The Winner is?

<u>Pairwise Comparison (or Condorcet Method)</u>

Even though the plurality method gives an appropriate winner, let’s see who prevails in pairwise contests. Revote for your preference (ex. Soda > Beer > Wine) No cheating: keep your first place vote from above. You have six choices; please circle your choice:

Soda > Beer > Wine   Beer > Soda > Wine   Wine > Soda > Beer

Soda > Wine > Beer   Beer > Wine > Soda   Wine > Beer > Soda

Next please fill in the following table for all people participating today. This table records the number of votes for each of the six preferences and then goes on to compare how each individual beverage fared against the other two in preference (pairwise).

Preferences	# of votes		Soda	Wine		Soda	Beer		Wine	Beer
S > W > B										
S > B > W										
W > S > B										
W > B > S										
B > S > W										
B > W > S										
TOTALS										

Who wins…. Soda vs. Wine?

Soda vs. Beer?

Wine vs. Beer?

Is there a clear winner beverage, i.e. a beverage that beats every other beverage?

<u>Some Disturbing Examples</u>

We will provide some data for examples which may cause concern. First we will limit the preferences to the three listed below. We will provide the number of votes for each preference, then you may conduct the pairwise comparisons to find the Condorcet winner.

Example #1

Preferences	# of votes		Soda	Wine		Soda	Beer		Wine	Beer
S > W > B										
B > W > S										
W > B > S										
TOTALS										

Who is the Plurality winner?

Who is the Condorcet winner?

Example #2: The Spoiler

Let's switch from beverages to politics (Bush, Kerry, Nader: B, K, N respectively). Again we will limit our example to three preferences, and provide you with the data. Please feel free to complete the pairwise comparisons.

Preferences	% of votes		Kerry	Nader		Kerry	Bush		Nader	Bush
K > N > B	498		498			498			498	
B > N > K	499			499			499			499
N > B > K	3			3			3		3	
TOTALS	1000									

Winner: **Winner:** **Winner:**

Who is the Plurality winner?

Who is the Condorcet winner?

Example #3

We will switch back to beverages for this example, still limited to only three preferences. Again we will provide the data, but feel free to complete the pairwise comparison.

Preferences	# of votes		Soda	Wine		Soda	Beer		Wine	Beer
S > W > B										
W > B > S										
B > W > S										
TOTALS										

Who is the Plurality winner?

Who is the Condorcet winner?

The Geometry of Condorcet Voting

If we only have three of the six possible preferences and

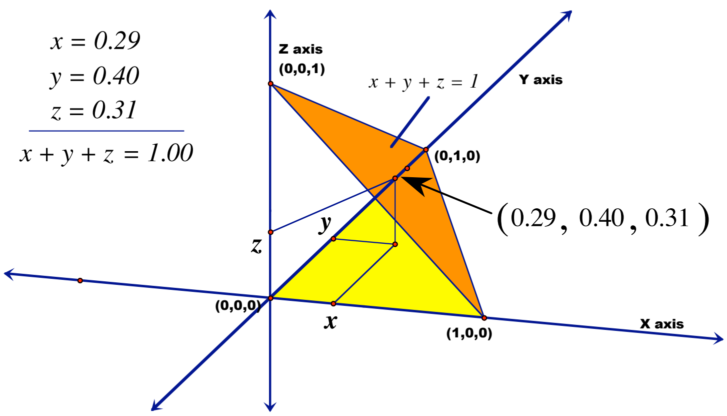
$$x = \frac{\text{\# of votes for preference 1}}{\text{number of voters}},$$

$$y = \frac{\text{\# of votes for preference 2}}{\text{number of voters}},$$

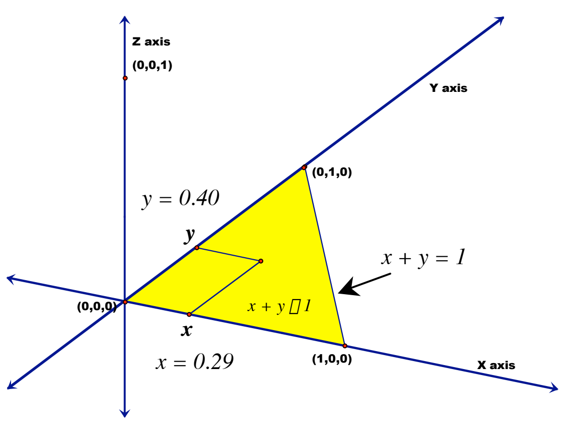
$$z = \frac{\text{\# of votes for preference 3}}{\text{number of voters}},$$ <u>and</u> $x \geq 0$, $y \geq 0$, $z \geq 0$,

then… $x + y + z = 1$.

This is the graph of a plane (represented by the darker triangle) in the first octant:



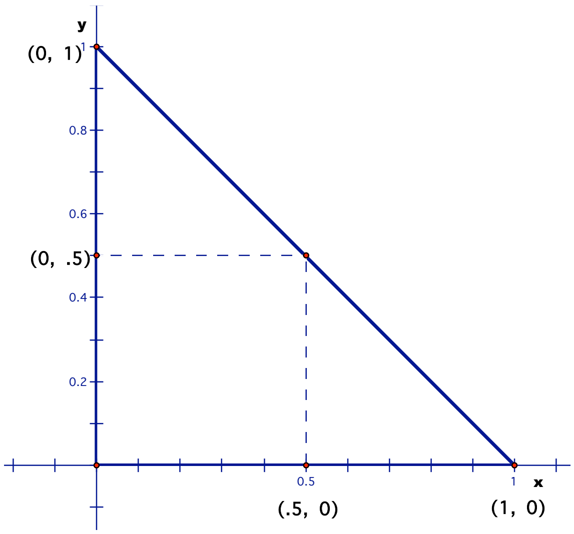
Since $z = 1 - x - y$, z is determined by x and y. Since $0 \leq x$, $0 \leq y$ and $x + y \leq 1$, we can restrict attention to the triangle in the xy plane:



Recall Preference 1: S > W > B with $x = \dfrac{\text{\# of votes for preference 1}}{\text{number of voters}}$

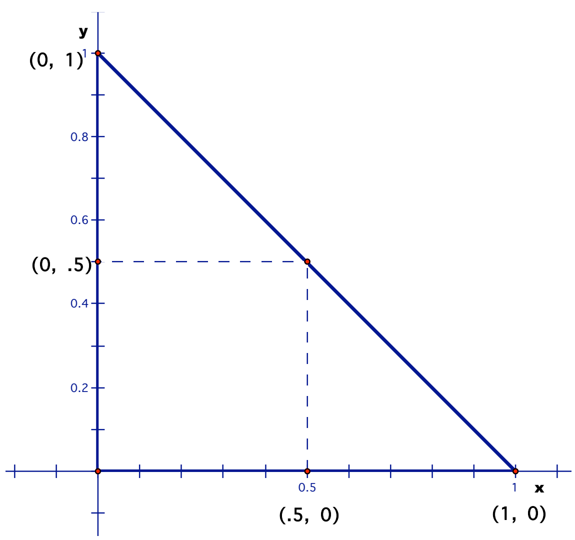
Preference 2: W > B > S with $y = \dfrac{\text{\# of votes for preference 2}}{\text{number of voters}}$

Prerference 3: B > S > W with $z = \dfrac{\text{\# of votes for preference 3}}{\text{number of voters}}$



In which region does Preference 1 win majority?

In which region does Preference 2 win majority?



Draw the boundary line of the region where Preference 3 wins a majority.
A **cyclic outcome** occurs when no preference beats both of the others. If the outcome is cyclic, where is (x,y) in this triangle?

Points in the Center Triangle Always Give Cyclic Outcomes

Let's say, as on page 6, we have the following three preferences:

> x is the fraction of voters who chose Preference 1: S > W > B
> y is the fraction of voters who chose Preference 2: W > B > S
> z is the fraction of voters who chose Preference 3: B > S > W

If (x, y) is in the middle triangle, then $x + y > .5$, $x < .5$, and $y < .5$. This means Wine beats Beer because Wine beats Beer in Preferences 1 & 2 (x & y) and, since $x + y > .5$ and $z < .5$, Beer can't muster enough votes to beat Wine.

How about Soda vs. Wine in the middle? Soda beats Wine in both x and z…

$$\begin{aligned} x + z &= x + (1 - x - y) \\ &= 1 - y \end{aligned}$$

Since y < .5, then $1 - y > .5$ which means $x + z > .5$ (Soda beats Wine).

Finally, Beer & Soda.

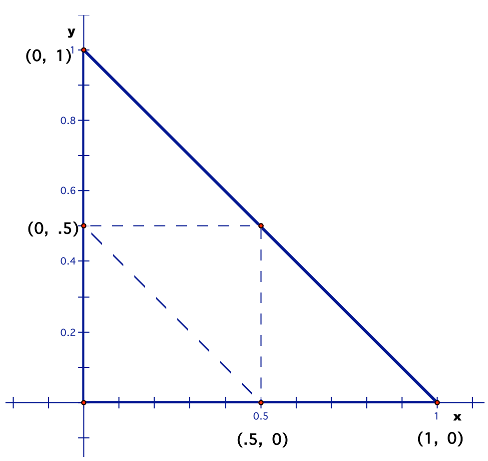
$$\begin{aligned} y + z &= y + (1 - x - y) \\ &= 1 - x \end{aligned}$$

Since $x < .5$, then $1 - x > .5$ which means $y + z > .5$ (Beer beats Soda).

This means W beats B ($x + y > .5$), S beats W ($x + z > .5$) and B beats S ($y + z > .5$).

So… we get a cyclic outcome whenever (x, y) is in the middle triangle.

What percent of time do we expect a cyclic outcome?



The Olympics and Weighted Voting

Spain, Romania and Hungary had similar medal counts from this past summer Olympics in Athens. Which country really finished fourteenth?

ATHENS 2004	Gold Medals	Silver Medals	Bronze Medals
Spain	3	11	5
Romania	8	5	6
Hungary	8	6	3

Who receives the highest total if Gold medals are worth one point, while Silver and Bronze medals are worth zero points? Let (G,S,B) = (1,0,0)

What if Gold medals are worth 2 points, Silver medals are worth 1 point and Bronze medals are worth 0 points, (G,S,B) = (2,1,0)?

Who receives the highest total?

What if Gold medals are worth 4 points, Silver medals are worth 2 points and Bronze medals are worth 0 points, (G,S,B) = (4,2,0)?

Who receives the highest total?

What do you notice?

What if Gold medals are worth 10 points, Silver medals are worth 9 points and Bronze medals are worth 1 point, (G,S,B) = (10,9,1)?

Who receives the highest total?

Weighted Voting or Borda Counts

In a Borda count, the first choice of a voter receives more points than the second choice, which receives more points than the third choice, etc. For example, we could give the first choice 3 points the second choice 2 points, the third choice 1 point and 0 points for the remaining choices. The winner is the choice which receives the most total points. We will use the notation (3,2,1) for this Borda count situation.

Think of Archery, Badminton, etc. as voters. Since South Korea, Taiwan and Ukraine won Gold, Silver and Bronze (respectively) in Archery, Archery gives South Korea 3 points, Taiwan 2 points, Ukraine 1 point, and all other countries 0 points. After all the Olympic sports "vote", the winner is the country with the most points. Therefore different Borda voting schemes could lead to different winners.

Borda Count for Beer, Wine, Soda

A voting vector (3, 2, 1) always gives the same outcome as the voting vector (6, 4, 2) or (9, 6, 3) or (12, 8, 4). (Why?) So (3, 2, 1) gives the same outcome as (1, 2/3, 1/3). This means we might as well assume that every voting vector is of the form (1, □, □), where □, □ are real numbers with 1 ≥ □ ≥ □ ≥ 0.

Consider the following preference table from Example 1:

Preference	# of votes	Points for S	Points for W	Points for B
S>W>B				
B>W>S				
W>B>S				

Totals: S: W: B:

To fill in the table, assume that □ = 0 (sorry, third choice). Pick a value of □ in {0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0}. Compute the Borda winner. Compare your answers for different values of □.

Note: If □ = 0, we have a "winner take all" or plurality vote.
If □ = 1, we have a "vote the loser off the island" vote.

Computing Borda Totals For a Fixed □

Recall that we have

x is the fraction of voters who prefer S>W>B,
y is the fraction of voters who prefer W>B>S,
z is the fraction of voters who prefer B>S>W.

So $x + y + z = 1$.

Consider a (1, □, 0) voting vector. Then Wine receives (a multiple of) □ $x + 1 \bullet y + 0 \bullet z$ = □ $x + y$ points, as the "*x*" voters rank Wine second (□), the "*y*" voters rank Wine first (1), and the "*z*" voters ranked Wine last (0).

Soda receives $1 \bullet x + 0 \bullet y +$ □ $z = x +$ □ $(1 - x - y)$ = (1 - □)x + □ $(1 - y)$ points. (Why? – Recall that $x + y + z = 1$.)

Complete the following table:

Beverage	Points
Soda 	(1 - □)x + □ $(1 - y)$
Wine 	□ $x + y$
Beer 	

The Geometry of the (1, □, 0) Winner

On the previous page, Soda beats Wine if (Soda pts) > (Wine pts). From the last page, this is

$(1-□)x + □(1-y) > □x + y$

and Wine beats Soda if

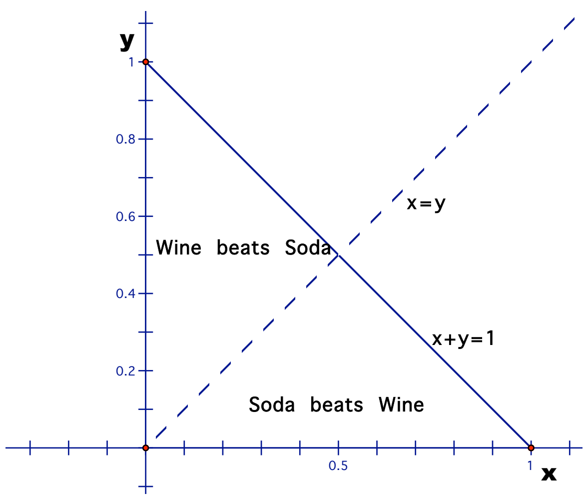
$(1-□)x + □(1-y) < □x+y$.

Wine <u>ties</u> Soda if

$(1-□)x + □(1-y) = □x+y$.

Let's say □=0 (Winner take all). Then we have a Soda-Wine tie if $x = y$ (Why?).

So, **Soda beats wine iff $x > y$.**

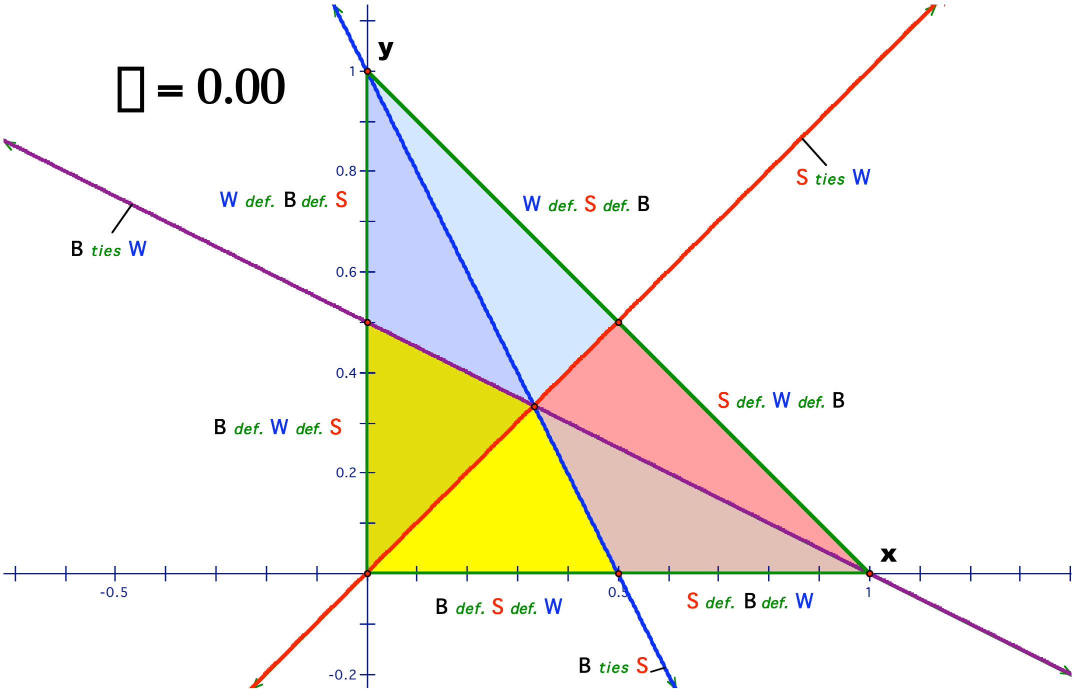


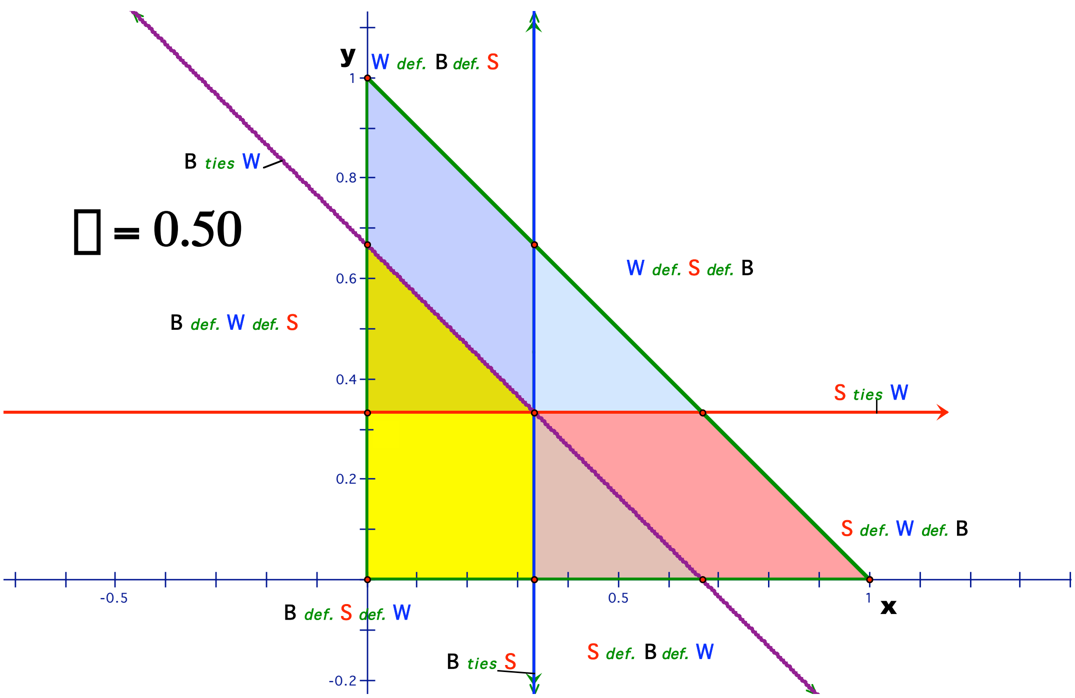
For general □, Soda and Wine tie when $(1-□)x + □(1-y) = □x + y$, or $(1-2□)x - (1+□)y = -□$.

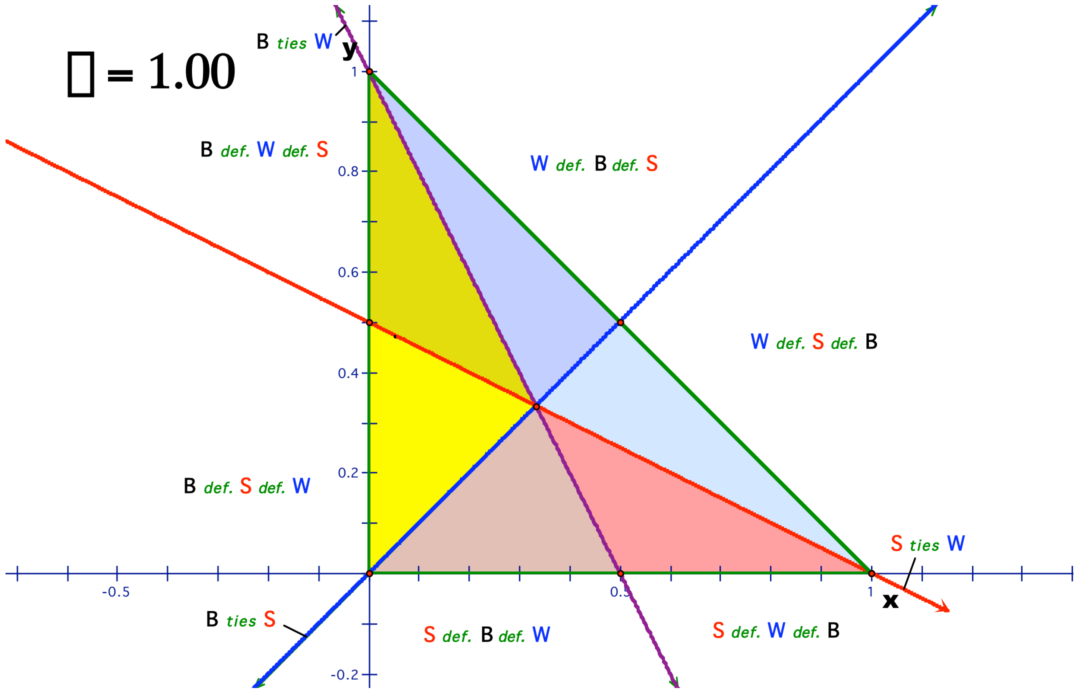
Find the other "tie lines."

Pair	Tie Line	Tie Line for □=0	Tie line for □=.5
Soda ties Wine	$(1-2□)x - (1+□)y = -□$.	$x = y$	$y = 1/3$
Beer ties Soda			
Wine ties Beer			

The Geometry of the (1, □, 0) Winner
Looking at different values of □







□ = 1.00
B ties W
y
x
B def. W def. S
W def. B def. S
W def. S def. B
B def. S def. W
S ties W
B ties S
S def. B def. W
S def. W def. B
1
0.8
0.6
0.4
0.2
-0.2
-0.5
0.5
1

Gaming the System

The polls predict the following outcome for the November 2, 2004, Beer/Wine/Soda faceoff:

36% of the voters have the preference W>B>S.
35% of the voters have the preference B>S>W.
29% of the voters have the preference S>W>B .

Your job as Chief Voting Official is to pick an equitable for the (1, , 0) voting vector.

Two nights before the election, the Beer Drinkers of America Friendship Group, which may have ties to the beer industry, offers you $1,000,000 to pick a so that Beer wins. Can you find a so that Beer wins just by a little, to avoid suspicion?

No problem: you want Beer to beat Wine. Beer is expected to receive .36• + .35 points (Why?), while Wine is expected to receive .36 + .29• points. So all you have to do is solve

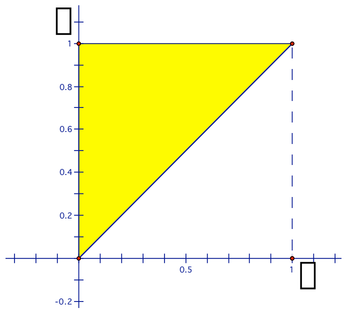
.36• + .35 > .36 + .29•

So for which values will Beer beat Wine? What value of will you select so that Beer just beats Wine?

The night before the election, you are approached by a representative of the Massachusetts Society of Chablis Sippers, who offers you $2,000,000 if Wine beats Beer by just a little. How should you adjust so that you get the extra million without raising the Beer Drinkers' suspicions?

The Feel Good Voting Scheme

As we have introduced earlier, the Bronze medal country receives more points than not medaling at all. This is what we refer to as a feel good score, where in a three choice race even last place receives some points. Let's look at the voting scheme where the country that receives the gold has 4 points, silver receives 2 points and 1 point for bronze. We will denote this as (4,2,1). Our next step is to scale this voting scheme so that the gold medal country would receive 1 point, and the other countries receive fraction of points. In order to do this we will divide or (4,2,1) voting scheme by 4, the points received by the gold medal country. This will give us the (1, .5, .25) voting scheme, which is equivalent to the (4,2,1) voting scheme. By this method any voting scheme can be converted to a (1, □, □) voting scheme, where 1 is the point value giving for first, □ is the point value given for second and □ is the point value given for third.



The shaded area gives the possible values of (□, □) with 0 ≤ □ ≤ □ ≤ 1.

Question 1: What would happen if every medaling country received an extra point for every medal regardless of first second or third? How would that affect the overall point total and rank?

Question 2: What would happen is the point values given to reach medal was to double for gold, silver and bronze? How would that affect the point total and overall rank?

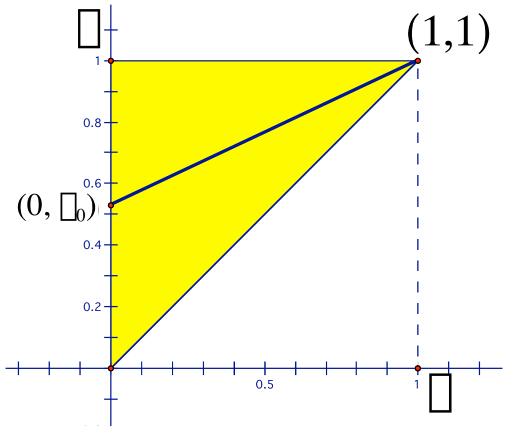
Let's say (4,2,1) is equivalent to (8,4,2) because for any distribution of votes, applying a Borda count of (4,2,1) yields the identical outcome as applying a Borda count of (8,4,2).

We know every Borda count is equivalent to a (1, □, □) voting vector. Is this the best we can do?

The Big Question: Are some (1, □, □) voting vectors equivalent to some other (1, □, □) voting vectors?

Equivalent (1, $\square$, $\square$) Voting Vectors

Why does the dark line consist of equivalent voting vectors?



1) (0, $\square_0$) represents the voting vector (1, $\square_0$, 0), which is equivalent to (A, A $\square_0$+1, 1) for any *A* $\square$ [0,1], by Question 2 on the last page.

2) (A, $\square\square_0$,0) is equivalent to (A+1, A $\square_0$+1, 1), by Question 1 on the last page. Again, by Question 1, (A+1, A $\square_0$+1, 1) is equivalent to (1, A $\square_0$+1-A, 1-A).

3) (1, A $\square_0$ +1-A, 1-A) is represented by the point (1-A, A $\square_0$ +1-A) in the shaded triangle. As A goes from 0 to 1, (1-A, A $\square_0$ +1-A) goes from (1,1) to (0, $\square_0$).

4) You check that the points (1-A, A $\square_0$ +1-A) all lie on the line from (0, $\square_0$) to (1,1). [Hint: write the equation of this line. Does (1-A, A $\square_0$ +1-A) satisfy this equation?]

The moral: Every voting vector (a, b, c) is equivalent to a unique voting vector of the form (1, $\square$, 0). So, psychological factors aside, you might as well give your third place choice zero points.

Food for thought

Question 1: What are the coordinates of the intersection of the tie lines? What is the significance of this point?

Question 2: On page 13 and 14, the three allowable preferences were S>W>B, W>B>S, B>S>W (see page 11). These preferences have a symmetry -- if we put S at 12 o'clock on a clock, the W at 4 o'clock and B at 8 o'clock, we get the three preferences by starting at one letter and reading off the three letters clockwise.

What if you choose three preferences that don't have this symmetry (e.g. the Bush-Kerry-Nader example)? Draw the tie lines. How do they change as lambda changes? Does the intersection of the three tie lines remain the same for all lambda?

Question 3: What if there are four candidates? How many possible outcomes are there? How would you keep track of all the possible outcomes? In a tetrahedron? What would be the generalization of the tie lines?

Question 4: What if there were 140 candidates (as in the California gubenatorial recall election)?