Investigating The Simplex Lock: A Mathematical Adventure

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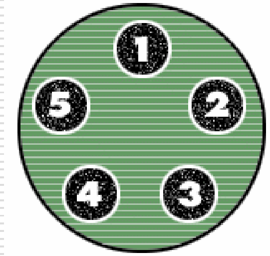
The Simplex Lock

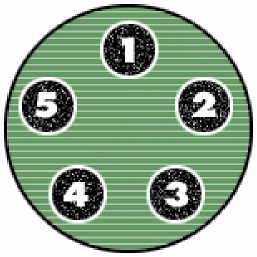
- 5-Button mechanical device
- Advertising claims “Thousands of combinations”
- How many are there?
- Let’s investigate!

Simplex Lock Combinations



- A combination is a sequence of 0–5 pushes using 0-5 buttons
- Each button is used at most once (once pushed, it stays in)
- Each push may include any of the buttons that have not yet been pushed, up to/including all remaining buttons
- A combination need *not* include all buttons ("unlocked" = 0 buttons, 0 pushes)
- When 2 or more buttons are pushed simultaneously, order does *not* matter

Combination Terminology

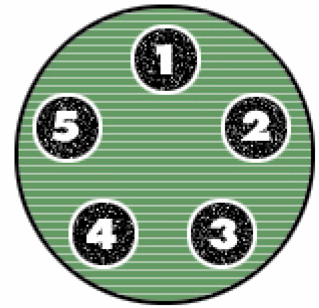


- "|" delineates buttons pushed in sequence
  - "1|2|3" uses 3 buttons, in order, 3 pushes
  - "3|2|1" is a different combination
- "+" groups buttons pushed simultaneously
  - "1+2+3" uses 3 buttons simultaneously, 1 push
  - Order not a factor: "1+2+3" = "3+2+1"
- "1|2+3" uses 3 buttons, 2 pushes
  - Push 1, then 2+3 simultaneously
  - "1|2+3" = "1|3+2"
  - "2+3|1" is a different combination

Examples of Simplex Lock Combinations

- “1+5|2|3|4” (5 buttons, 4 pushes)
- “2|1+5|3|4” (5 buttons, 4 pushes)

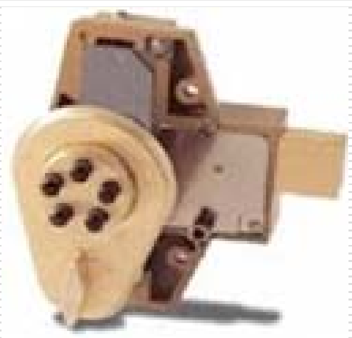
- “1+2+3+4+5” (5 buttons, 1 push)



- “1|2|3|4|5” (5 buttons, 5 pushes)
- “2|1|3|4|5” (5 buttons, 5 pushes)

- “4+2” (2 buttons, 1 push)
- “4+5” (2 buttons, 1 push)
- “1+3|4+5” (4 buttons, 2 pushes)

Comparing Mechanical Locks

Comparison of Locks and Their Numbers of Combinations		
		
5-Button Simplex Lock	Master Combination Lock (3 turns, 40 positions each*)	Barrel Combination Lock (4 wheels, 10 positions each)
???	$40^3 \approx 64{,}000*$	$10^4 = 10{,}000$

Problem Solving Strategies

- Simplify
  - Start with a simpler problem
- Organize
  - Tables
  - Enumeration strategies; count by:
    - Buttons used (0-5)
    - Pushes used (0-5)
    - Shape (xxx, x|xx, xx|x, x|x|x)
- Generalize

Strategy: Simplify

- List the combinations on a 1-button Simplex Lock
  - Unlocked (0 buttons, 0 pushes)
  - 1 (1 button, 1 push)

- Now, list the combinations on a 2-button Simplex Lock

Strategy: Simplify & Organize

- Combinations on a 2-button Simplex Lock
  - Unlocked (0 buttons, 0 pushes)
  - 1 (1 button, 1 push)
  - 2 (1 button, 1 push)
  - 1+2 (2 buttons, 1 push)
  - 1|2 (2 buttons, 2 pushes)
  - 2|1 (2 buttons, 2 pushes)
- Now, list the combinations on a 3-button Simplex Lock

Factorial Notation

- *n*! is the product of all positive integers less than or equal to *n*
- 3! = 3 x 2 x 1
  - A set of 3 elements can be *arranged* 6 ways
- 1! = 1
- Recall that 0! = 1
  - For students who are confused by this, I remind them that there is still 1 way to arrange an empty set of elements (see 0-button lock)

Combinatorics: Permutations

- Permutation: ordered *arrangement* of elements from a set
- In how many ways can a set of 3 elements be *arranged*?
  - List them
  - Any ways to count these?

Combinatorics: Permutations

- $P(n,r)$ is the number of possible arrangements of *r* elements from a set of *n* elements

$$P(n,r)=\frac{n!}{(n-r)!}$$

- On a 3-button lock, the number of *sequential* 2-button arrangements (example: “1|2”) is $P(3,2)$

$$P(3,2)=\frac{3!}{(3-2)!}=6$$

- On a 3-button lock, the number of *sequential* 3-button arrangements (example: “1|2|3”) is $P(3,3)$

$$P(3,3)=\frac{3!}{(3-3)!}=6$$

Combinatorics: Combinations

- "*n* choose *k*" is the number of ways to choose *k* elements (combinations) from a set of *n* elements

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

- On a 5-button lock, the number of 3-button, 1-push combinations (example: "2+5+3") is "5 choose 3"

$$\binom{5}{3} = \frac{5!}{2!3!} = 10$$

Combinatorial Notation

- On a 5-button lock, "1|2+3" is "5 choose 1, 4 choose 2"

$$\binom{5}{1}\binom{4}{2} = \left(\frac{5!}{4!1!}\right)\left(\frac{4!}{2!}\right) = \frac{5!4!}{4!1!2!} = 30$$

- On a 5-button lock, "1+2|3" is "5 choose 2, 3 choose 1"

$$\binom{5}{2}\binom{3}{1} = \left(\frac{5!}{3!2!}\right)\left(\frac{3!}{2!1!}\right) = \frac{5!3!}{3!2!2!1!} = 30$$

- Since these behave identically, we simplify counting of permutations such as "1|2+3" and "1+2|3" as follows

$$2\binom{5}{1}\binom{4}{2} = 2(30) = 60$$

Choose Numbers in Pascal's Triangle

- For "*n* choose *k*" find row *n*, entry *k*
  - The first row is numbered 0
  - The first entry in a row is numbered 0

0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

- "4 choose 2" = 6

3-Button Lock: Organize (by # buttons)

<table>
<tr><th>Buttons used (k)</th><th>Example</th><th>Notation</th><th>$\Sigma(k)$</th><th>Total</th></tr>
<tr><td>0</td><td>Unlocked</td><td>$\binom{3}{0}=1$</td><td>1</td><td rowspan="7">26</td></tr>
<tr><td>1</td><td>1 or 2 or 3</td><td>$\binom{3}{1}=3$</td><td>3</td></tr>
<tr><td rowspan="2">2</td><td>1|2</td><td>$\binom{3}{1}\binom{2}{1}=6$</td><td rowspan="2">9</td></tr>
<tr><td>1+2</td><td>$\binom{3}{2}=3$</td></tr>
<tr><td rowspan="3">3</td><td>1|2|3</td><td>$\binom{3}{1}\binom{2}{1}\binom{1}{1}=6$</td><td rowspan="3">13</td></tr>
<tr><td>1|2+3
1+2|3</td><td>$2\binom{3}{1}\binom{2}{2}=2(3)=6$</td></tr>
<tr><td>1+2+3</td><td>$\binom{3}{3}=1$</td></tr>
</table>

3-Button Lock: Organize (by # pushes)

<table>
<tr><th># Pushes</th><th>Example</th><th>Notation</th><th>$\Sigma(\textit{pushes})$</th><th>Total</th></tr>
<tr><td>0</td><td>Unlocked</td><td>$\binom{3}{0}=1$</td><td>1</td><td rowspan="7">26</td></tr>
<tr><td>1</td><td>1 or 2 or 3</td><td>$\binom{3}{1}=3$</td><td rowspan="3">7</td></tr>
<tr><td></td><td>1+2</td><td>$\binom{3}{2}=3$</td></tr>
<tr><td></td><td>1+2+3</td><td>$\binom{3}{3}=1$</td></tr>
<tr><td>2</td><td>1|2</td><td>$\binom{3}{1}\binom{2}{1}=6$</td><td rowspan="2">12</td></tr>
<tr><td></td><td>1|2+3
1+2|3</td><td>$2\binom{3}{1}\binom{2}{2}=2(3)=6$</td></tr>
<tr><td>3</td><td>1|2|3</td><td>$\binom{3}{1}\binom{2}{1}\binom{1}{1}=6$</td><td>6</td></tr>
</table>

The 5-Button Simplex Lock

***5*-button lock using *0* buttons**	
Shapes	**# Combinations**
0	1 (unlocked)
***5*-button lock using *1* button**	
Shapes	**# Combinations**
1	$\binom{5}{1} = 5$

The 5-Button Simplex Lock

***5*-button lock using *2* buttons**	
Shapes	**# Combinations**
1|2	$\binom{5}{1}\binom{4}{1} = 20$
1+2	$\binom{5}{2} = 10$

The 5-Button Simplex Lock

***5*-button lock using *3* buttons**	
Shapes	**# Combinations**
1|2|3	$\binom{5}{1}\binom{4}{1}\binom{3}{1}=60$
1|2+3 1+2|3	$2\binom{5}{1}\binom{4}{2}=2(30)=60$
1+2+3	$\binom{5}{3}=10$

The 5-Button Simplex Lock

***5*-button lock using *4* buttons**	
Shapes	**# Combinations**
1|2|3|4	$\binom{5}{1}\binom{4}{1}\binom{3}{1}\binom{2}{1}=120$
1|2|3+4 1|2+3|4 1+2|3|4	$3\binom{5}{1}\binom{4}{1}\binom{3}{2}=3(60)=180$
1+2|3+4	$\binom{5}{2}\binom{3}{2}=30$
1|2+3+4 1+2+3|4	$2\binom{5}{1}\binom{4}{3}=2(20)=40$
1+2+3+4	$\binom{5}{4}=5$

The 5-Button Simplex Lock

# Buttons Used	# Shapes	# Combinations
0	0	1
1	1	5
2	2	30
3	4	130
4	8	375
5	?	???

The 5-Button Simplex Lock

- Find numbers of combinations
- Investigate the patterns
- Make conjectures & generalizations
  - Shapes for *n*-button combinations
  - Combinations using *n* buttons
  - Combinations using *k* pushes
- Prove ‘em
- Are there formulas?
- Excel, MATLAB etc.
- Extensions

Simplex Lock Extensions

- In the 1960's, German-born mathematician Kurt Mahler introduced a form of polynomial expansions of binomial coefficients of the form $\binom{x}{n}$ where *n* are positive integers:

$$\binom{x}{1} = \frac{x!}{(x-1)!1!} = \frac{x}{1} = x$$

$$\binom{x}{2} = \frac{x!}{(x-2)!2!} = \frac{(x)(x-1)}{2!} = \frac{x^2 - x}{2}$$

$$\binom{x}{3} = \frac{x!}{(x-3)!3!} = \frac{(x)(x-1)(x-2)}{3!} = \frac{(x-2)(x^2-x)}{6} = \frac{x^3 - 3x^2 + 2x}{6}$$

$$\binom{x}{4} = \frac{x!}{(x-4)!4!} = \frac{(x)(x-1)(x-2)(x-3)}{4!} = \frac{(x-3)(x^3-3x^2+2x)}{24} = \frac{x^4 - 6x^3 + 11x^2 - 6x}{24}$$

$$\binom{x}{5} = \frac{x!}{(x-5)!5!} = \frac{(x)(x-1)(x-2)(x-3)(x-4)}{5!} = \frac{(x-4)(x^4-6x^3+11x^2-6x)}{120} = \frac{x^5 - 10x^4 + 35x^3 - 50x^2 + 24x}{24}$$

Simplex Lock Extensions

- Writing each of these Mahler expansions in terms of relative powers of *x* we have:

$$x^1 = 1\binom{x}{1}$$

$$x^2 = 2\binom{x}{2} + 1\binom{x}{1}$$

$$x^3 = 6\binom{x}{3} + 6\binom{x}{2} + 1\binom{x}{1}$$

$$x^4 = 24\binom{x}{4} + 36\binom{x}{3} + 14\binom{x}{2} + 1\binom{x}{1}$$

$$x^5 = 120\binom{x}{5} + 240\binom{x}{4} + 150\binom{x}{3} + 30\binom{x}{2} + 1\binom{x}{1}$$

Simplex Lock Extensions

- Comparing the 3rd Mahler expansion with the combinations of a 3-Button Simplex Lock *that utilize all 3 buttons*, we see some interesting connections:

$$x^3 = 6\binom{x}{3} + 6\binom{x}{2} + 1\binom{x}{1}$$

3	1|2|3	$\binom{3}{1}\binom{2}{1}\binom{1}{1} = 6$
	1|2+3 1+2|3	$2\binom{3}{1}\binom{2}{2} = 6$
	1+2+3	$\binom{3}{3} = 1$

My Own Current Research

- Closed form Simplex Lock formula?
- Comparing Simplex Lock and Mahler basis
- Proving connection between Mahler basis and Simplex Lock combinations
- Investigating “Pascal-like” triangles generated by Simplex Lock combinations and Mahler basis terms

The Simplex Lock Problem

- Real world problem
- Open-ended investigation
- Entry at various skill levels
- Cooperative learning & sharing
- Multiple levels of exploration at various levels of interest, skill & time

The Simplex Lock Problem

- Thanks to Al Cuoco at EDC, and to Brian Harvey at UC Berkeley
- Special thanks to Professor Robert Pollack at Boston University
- More info on this and other cool projects at EDC website:

http://www2.edc.org/makingmath/mathproj.asp

- Contact me at *adkatz@bu.edu*