The Moduli Space of Triangles and Heron Triangles

Steve Rosenberg (Boston U.), Mike Spillane (Watertown HS), Dan Wulf (Watertown HS)

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Warmup: What do the triangles with sides $20, 13, 11$ and with sides $6593350940, 4475827709, 3456855291$ have to do with each other?

The Focus on Math MSP brings together mathematicians from three universities and grades 4-12 math teachers from five school districts to build a community of mathematics learners and researchers – everybody learns, everybody does research. Our study group meets every two weeks. We developed a three hour seminar on the Geometry of Voting, which we presented at the October 2005 NCTM meeting. Last year, we focused on an elementary geometry problem posed by the Watertown HS math district leader.

The Motivating Problem

Are two triangles with the same perimeter and same area congruent?

Using Geometer's Sketchpad, we quickly saw that the answer is no. We did note via Sketchpad that triangles with the same perimeter and area have the same incircle.

Main theme: Geometer's Sketchpad leads to many conjectures from the high school teachers and allows us to consider *all triangles at once.*

The Algebraic Approach

Our first step was to find a way to express area in terms of perimeter or triangle sides. We fixed one side of a triangle to be $c \in (0, s)$, where s is the semiperimeter. (Each side of a triangle must be less than s by the triangle inequality.) We call the other two sides of the triangle a and b. Since $2s = a + b + c$, we have $b = 2s - a - c$.

Fix the semiperimeter s, area A and side c. Heron's formula is

$$A^2 = s(s-c)(s-a)(s-(2s-a-c)).$$

(This uses the Law of Cosines.)

This is a (messy) quadratic equation for a, which gives

$$a = \frac{2s-c}{2} \pm \left(\frac{c^2}{4} - \frac{A^2}{s(s-c)} \right)^{1/2}.$$

(b is given by choosing the "other sign.") We want the roots to be real, so we need

$$\frac{c^2}{4} - \frac{A^2}{s(s-c)} \geq 0,$$

or equivalently

$$-sc^3 + s^2c^2 - 4A^2 \geq 0.$$

Summary: For fixed A, s, if the *evil cubic*

$$f(c) = -sc^3 + s^2c^2 - 4A^2$$

is nonnegative for some $c > 0$, then there exists a *unique* triangle with side c, area A, semiperimeter s.

By calculus, $f(c)$ is negative at $c = s$, has a negative minimum at $c = 0$ and a maximum at $(\frac{2}{3}s, \frac{4}{27}s^4 - 4A^2)$. Thus the evil cubic is positive for some positive $c \in (0, s)$ if $\frac{4}{27}s^4 - 4A^2 \geq 0$, or

$$A^2 \leq \frac{s^4}{27}.$$

So a triangle with semiperimeter s and area A exists iff this inequality holds. We checked that we get equality precisely at the equilateral triangle. (So this solves the triangle isoperimetric problem: the equilateral triangle maximizes area among all triangles of fixed perimeter. This is usually a multivariable calc problem.)

Summary: Fix A, s with $A^2 \leq \frac{s^4}{27}$. Then for any c lying between the positive roots r_1 and r_2 of the evil cubic, there exists a *unique* triangle with area A, semiperimeter s, and side c. So for these A, s, there exist *infinitely* many noncongruent triangles with the same A and s (except for the equilateral case, where $r_1 = r_2$).

Problem: This answers our original question, but we can't find the roots of the evil cubic explicitly.

(SR: Welcome to research mathematics.)

The Geometric Approach

First simplification: Fix a scale by assuming e.g. $A = 1$ or $s = 1$.

Second simplification: Pick good parameters. From experience, the best parameters are: set $s = 1$ and use c, A as parameters. So triangles with $s = 1$ correspond to points in the (c, A)-plane.

From now on, all triangles have $s = 1$ unless otherwise stated.

Using Heron's formula and a little (avoidable) calculus, for fixed c the maximum A occurs when $a = b$ (isosceles triangle), and the maximum is at

$$(\star) \qquad A = \frac{c}{2}\sqrt{1-c}$$

So triangles correspond to points in the (c, A)-plane in the first quadrant and under this curve.

Interpretation of the evil cubic: a horizontal line at A_0 cuts the graph of $(\star)$ at two points (r_1, A_0) and (r_2, A_0) where r_1, r_2 are the roots of the evil cubic $f(c) = -c^3 + c^2 - 4A_0^2$.

The set of triangles with area A_0 correspond to points on the horizontal line between (r_1, A_0) and (r_2, A_0).

However, when we travel across a horizontal line, we see the same triangle three times.

This is because isosceles triangles can occur when $a = b$, $a = c$, or $b = c$. These last two possibilities occur when

$$A = (1 - c)\sqrt{2c - 1}$$

(Heron again).

Adding this curve to our figure, we get three regions, I, II, III. In each region, a point corresponds to a unique triangle with $s = 1$, and every such triangle corresponds to a unique point in the region. No two points in a region correspond to congruent *or even similar* triangles. Mathematicians call such a region a *moduli space*, in this case for the set of scaled triangles. But who cares about the terminology?

The boundary curves for each region label all isosceles triangles, and note the special point corresponding to the equilateral triangle.

The Geometry of the Moduli Space

A. We know *horizontal lines* cut across the moduli space at triangles of equal area.

B. *Vertical lines* cut across the moduli space in triangles whose top vertex C lies on an ellipse with foci at the ends of the c segment.

C. Using Geometer's Sketchpad, one HS teacher conjectured and proved that straight lines emanating from $(1, 0)$ cut across region III at triangles with constant angle at vertex C.

There are many interesting lines and curves in the moduli space waiting to be explored.....

Heron Triangles: Bringing in Number Theory

A *Heron triangle* is a triangle with integer sides and integer area. A triangle with rational sides and rational area yields a Heron triangle after rescaling, so we can just look for triangles with rational sides and area. In particular, we can demand $s = 1$ and look for rational Heron triangles inside our moduli space

Example: (i) Right triangles with integer sides (Pythagorean triples). These are given by $(a^i, b^i, c^i) = (2i, i^2-1, i^2+1)$ for $i = 2, 3, ...$ (Almost) all of these have different scale-free ratios A/s^2. The $s = 1$ scaled versions are

$$\left(\frac{2i}{i(i+1)}, \frac{i^2-1}{i(i+1)}, \frac{i^2+1}{i(i+1)}\right)$$

with areas $A^i = \frac{i-1}{i(i+1)}$.

(ii) The triangle $(20, 13, 11)$ has area 66 and is not a right triangle. The $s = 1$ version is $(20/22, 13/22, 11/22)$.

Where are the Heron Triangles in our Moduli Space?

A state-of-the-art research result:

Theorem: (van Luijk, `math.AG/0411606`) *For any integer N, there exists an infinite sequence of N Heron triangles*

$$(a_1^i, b_1^i, c_1^i), (a_2^i, b_2^i, c_2^i), \ldots, (a_N^i, b_N^i, c_N^i),$$

for $i = 1, 2, 3, \ldots$ such that for fixed i, each of the N triangles has the same scale-free ratio $A^i/(s^i)^2$, with $A^i/(s^i)^2 \neq A^j/(s^j)^2$ for $i \neq j$. No two triangles in this N-fold infinite sequence are similar.

Interpretation of van Luijk's theorem in our moduli space: Fix a positive integer N. We want to find an infinite number of areas (heights) $A^i, i = 1, 2, 3, \ldots$ such that on each horizontal line $A = A^i$, there are N points $(c_1^i, A^i), \ldots (c_N^i, A^i)$ with both coordinates rational and with corresponding a and b lengths rational. This will give N Heron triangles with fixed "ratio" A^i (as $s = 1$), and all triangles will be non-similar.

van Luijk's theorem is from 2004. It was only in 2000 that the case $N = 2$ was solved. van Luijk's proof is very sophisticated, using algebraic geometry. We will take a crack at it using our moduli space.

Constructing Heron Triangles

Draw the line from $(0,1)$ to $(t,0)$ $(t>1)$. It hits the unit circle at $\left(\frac{2t}{t^2+1},\frac{t^2-1}{t^2+1}\right)$, so this point has rational coordinates precisely if $t\in\mathbb{Q}$. Let ψ be the corresponding angle, so $\cos(\psi)=\frac{2t}{t^2+1}$, $\sin(\psi)=\frac{t^2-1}{t^2+1}$. We'll write "$\psi\in\mathbb{Q}$" if $t\in\mathbb{Q}$ (equivalently, if $\cos(\psi),\sin(\psi)\in\mathbb{Q}$).

Consider ΔABC with sides a,b,c and fixed angle ψ at vertex B. (Before we fixed the angle at C.) By the Law of Cosines,

$$a=\frac{2(1-c)}{2-c(1+\cos(\psi))},\ b=2-a-c,\ A=\frac{1}{2}ac\sin(\psi)=\frac{c(1-c)\sin(\psi)}{2-c(1+\cos(\psi))}.$$

Therefore

$$c,\psi\in\mathbb{Q}\Longrightarrow a\in\mathbb{Q}\Longrightarrow A,b\in\mathbb{Q}\Longrightarrow\Delta ABC\text{ is Heron.}$$

Fix a rational angle ψ. Then

$$A_\psi(c)=\frac{c(1-c)\sin(\psi)}{2-c(1+\cos(\psi))}$$

is the area of a triangle with base c and base angle ψ.

By calculus, $A_\psi(c)$ is increasing for c near 0. So if c runs over an infinite number of small rational numbers $c_1, c_2, \ldots$ near zero, we get an infinite number of Heron triangles corresponding to the points $(c_1, A_\psi(c_1)), (c_2, A_\psi(c_2)), \ldots$ on the graph of A_ψ.

To prove van Luijk's theorem, "all" we have to do is to fix a height $A^i = A_\psi(c^i)$, and find $N - 1$ more Heron triangles at this height.

Why is van Luijk's Theorem Hard?

1. Fix a rational angle $\psi = \psi^1$. We have an infinite number of Heron triangles in region I lying on the graph of

$$A_\psi(c) = \frac{c(1-c)\sin(\psi)}{2 - c(1+\cos(\psi))}.$$

2. The key difficulty: Vary ψ slightly to another rational angle ψ^2. We get an infinite number of Heron triangles on the graph of $A_{\psi^2}(c)$. *But do any of the heights (= areas of the Heron triangles) for ψ^2 agree with the heights of the Heron triangles for ψ?*

Example: Say $\psi = \pi/2$. One triangle on $A_{\pi/2}(c)$ is $(3/5, 6/25)$, which corresponds to the (scaled down) (3,4,5) right triangle. Take ψ^2 so that e.g. $\cos(\psi^2) = 8/17, \sin(\psi^2) = 15/17$. Is there a $c \in \mathbb{Q}$ such that

$$6/25 = A_{\psi^2}(c) = \frac{c(1-c)\frac{15}{17}}{2 - c(1+\frac{8}{17})}$$

This is a quadratic equation in c, whose solution involves a square root. We have to be very lucky or clever to get $c \in \mathbb{Q}$.

Consolation Results

It's hard to find N Heron triangles on a horizontal line in our moduli space.

A. *We can find an infinite number of Heron triangles on any vertical line over* $c \in \mathbb{Q}$.

Just take all the rational $t > 1$. This produces an infinite number of rational angles ψ, and the points $(c, A_\psi(c))$ are Heron triangles on the vertical line. They are all distinct for small values of t.

In fact, as t increases, the triangles grow from a degenerate triangle up to the isosceles triangle with base c (and then sometimes "comes back down").

B. *The set of Heron triangles is dense in the moduli space.*

Since the rational $t > 1$ are dense in $(1, \infty)$, the set of Heron triangles is dense on any vertical line over $c \in \mathbb{Q}$ by A. Now let the c in part A range over all rational numbers in $(0, 1)$.

An Example of Close Heron Triangles

Take e.g. $t = 2$, so $\cos(\psi) = 4/5, \sin(\psi) = 3/5$. Using our formulas for a, b as functions of c, we get

$$(a(c), b(c), c) = \left(\frac{10(1-c)}{10-9c}, \frac{10-18c+9c^2}{10-9c}, c \right),$$

which is equivalent to $(10(1-c), 10-18c+9c^2, c(10-9c))$.

These triangles correspond to the points on the curve $A_\psi(c)$ in the moduli space.

For $c = 1/2$ we get the Heron triangle $(20, 13, 11)$ after scaling.

Step 1. List the rationals close to $c = 1/2$ as e.g. $c^1 = 1/2, c^2 = 7/16, c_3 = 17/32, \ldots$. After clearing denominators, we get an infinite sequence of Heron triangles

$$(20, 13, 11), (1440, 985, 679), (4800, 3049, 2839), \ldots$$

with $A^i = 66, 313698, 4088160, \ldots$ and with fixed angle at one vertex.

To proceed to get a close Heron triangle, we could fix c and let t vary, but for fun let's have both c and t vary.

Step 2. Pick e.g. $t' = 2.1 \approx t$, so the corresponding angle ψ^2 has $\cos(\psi_2) = 4.2/5.41, \sin(\psi_2) = 3.41/5.41$. This gives a second infinite sequence of Heron triangles

$$\left(\frac{10.82(1-c)}{10.82-9.61c}, \frac{10.82-19.22c+9.61c^2}{10.82-9.61c}, c\right)$$

Compare with our first sequence

$$(a(c), b(c), c) = \left(\frac{10(1-c)}{10-9c}, \frac{10-18c+9c^2}{10-9c}, c\right).$$

Rescale the Step 1 Heron triangles to have $s = 1$, so e.g. the first one is $(20/22, 13/22, 11/22)$ corresponding to $t = 2, c = 1/2$. We now have $t' = 2.1$, and let's take a new c, say $c' = 1623/3410 \approx c$.

This gives the first (scaled up integer) Heron triangle in the second sequence: $(6593350940, 4475827709, 3456855291)$. A mental check shows this has area 718314296495781770 and satisfies $a^2 + c^2 - 2ac(4.2/5.41) = b^2$. After rescaling to $s = 1$, this triangle corresponds to a point in the moduli space very close to the point for the scaled $(20, 13, 11)$ triangle, so these two triangles are *almost similar.*

Warmup: What do the triangles with sides $20, 13, 11$ and with sides $6593350940, 4475827709, 3456855291$ have to do with each other?

Answer: These Heron triangles have integer area and are almost similar.

Conclusions:

1. We took a problem in geometry, expressed it algebraically, and worked on it until we got stuck with an evil cubic.

2. We realized we had to keep track of all (scaled) triangles simultaneously, which we did by constructing a moduli space, a geometric object. The boundary curves of the moduli space are given by relatively simple algebraic equations. *So the geometry and the algebra of this problem are linked at many levels.*

3. Our moduli space is full of information. Isosceles and equilateral triangles cut out special curves in the moduli space, and horizontal and vertical lines have special interpretations. The number theory of Heron triangles appears as a dense subset of the moduli space, so we can produce integral Heron triangles which are nonsimilar, but arbitrarily close to being similar.

4. Our proofs and conjectures were based on examining the moduli space of all triangles with semiperimeter one through the use of Geometer's Sketchpad.

5. Most importantly, high school math teachers are capable of doing sophisticated, research level math. The HS teachers are active conjecture-makers and theorem-provers in this process.