MSP Learning Network Conference: Challenging Courses and Curricula
Tuesday, February 1, 2005 Break Out Session #8

Focus on Mathematics

Knowledge of Mathematics for Teaching: How to define it? How to assess it? How to achieve it?

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Focus on Mathematics

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Is mathematics a tool?

Part of the world?

Invented?

Is mathematics something one does?

Students should be *pattern sniffers*

Students should be *experimenters*

Students should be *describers*

Students should be *tinkerers*

Students should be *inventors*

Students should be *visualizers*

Students should be *conjecturers*

Students should be *guessers*

Elementary teachers should know basic principles and concepts in:

- *number sense and numeration*
- *patterns and functions*
- *geometry and measurement*
- *data analysis*

Middle and H.S. teachers should know:

- *algebra*
- *geometry*
- *trigonometry*
- *discrete math*
- *intro to calculus*
- *history of math*
- *use of technology*

H.S. teachers should additionally know:

- *abstract algebra*
- *number theory*
- *calculus*
- *prob. & statistics*
- *trans. geometry*
- *applied math*

All mathematics teachers must learn:

- ***to experiment (wander with ingenuity)***
- ***to use abstraction***
- ***to develop and use theories***
- ***to solve problems***
- ***to struggle and <u>not</u> solve problems***
- ***technical fluency***

All mathematics teachers must recognize the following qualities of mathematics:

- ***utility of mathematics***
- ***intrinsic beauty***
- ***historical value***

All mathematics teachers must develop a mathematical disposition.

<u>"Knowledge of Mathematics for Teaching"</u>

- **Knowing (?) mathematics *content***

- **Knowing which concepts are easy or difficult to learn and why**

- **Knowing ways of representing concepts so that others can understand them**

- **Knowing how to connect ideas to deepen them**

- **Recognizing what students might be thinking or understanding**

But the perspective that is too easy to miss, and might be the most critical, is:

- **Experience thinking (and struggling) as a mathematician does**

The Focus on Mathematics approach to "Mathematics for Teaching"

(1) ***Look at mathematics in the daily work of teaching***

(2) ***Look at the knowledge used by expert teachers***

(3) ***Design programs that develop this expertise***

(1) Look at the daily work of teaching

• The class is using calculators and estimation to get decimal approximations to $\sqrt{5}$. One student looks at how you do out long multiplication and realizes that none of these decimals would ever work, because if you square a finite (non-integer) decimal, there'll be a digit to the right of the decimal point. So you can't ever get an integer. She deduces that that $\sqrt{5}$ can't be rational. **If this happened in your class, what would you do?**

— adapted from "A Dialogue About Teaching" in *What's Happening in Math Class?* Teacher's College Press.

• Nine year old David, experimenting with numbers, conjectures that, if the period for the decimal expansion of $\frac{1}{n}$ is $n-1$, then n is prime. **Is David's conjecture valid? Is the converse true?**

— Adapted from a Reader Reflection by Walt Levisee in the *Mathematics Teacher* (March, 1997).

• Speaking of decimals, how would you characterize the "unit fractions" $\frac{1}{n}$ that have terminating decimal expansions? What can you say about the periods of the repeating ones? **How could your students approach these questions?**

- This problem causes no difficulty with prealgebra students who understand the connection between rate, time, and distance:

> *Mary drives from Boston to Washington, a trip of 500 miles. If she travels at and average of 60 MPH on the way down and 50 MPH on the way back, how many hours does her trip take?*

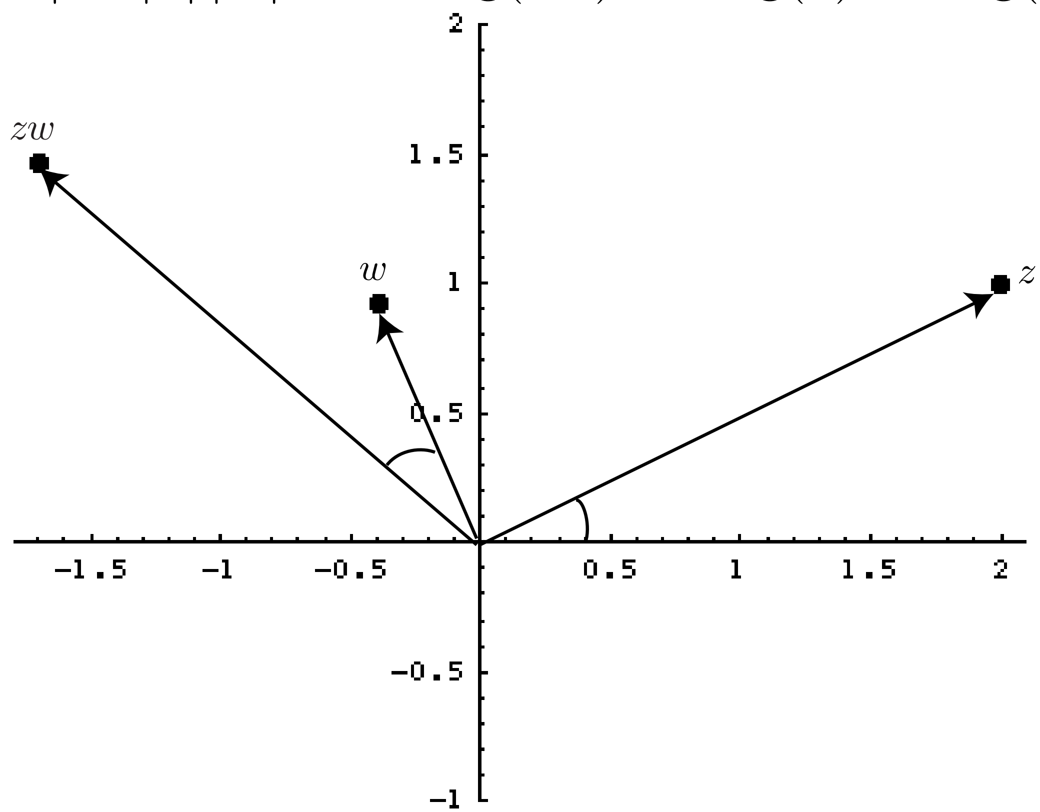
While *this* problem is baffling to many of the same students a year later in algebra class:

> *Mary drives from Boston to Washington, and she travels at and average of 60 MPH on the way down and 50 MPH on the way back. If the total trip takes* $18\frac{1}{3}$ *hours, how far is Boston from Washington?*

What's going on here?

• How would you help your students understand the "multiplication rule" for complex numbers?

$$|zw| = |z|\,|w| \text{ and } \mathrm{Arg}(zw) = \mathrm{Arg}(z) + \mathrm{Arg}(w)$$



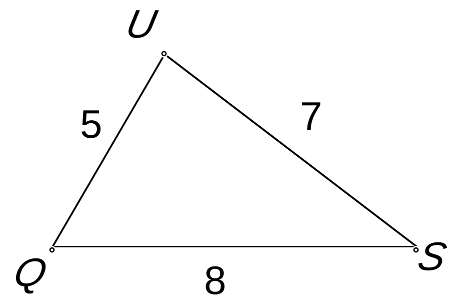
What if they didn't know any trig?

- How can you find a “simple” rule that agrees with this table?

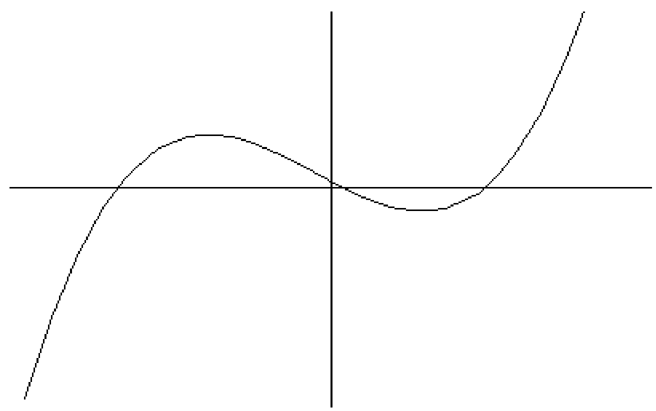
Input	Output
0	1
1	6
2	63
3	364
4	1365
5	3906
6	9331
7	19608

What’s known about this and related problems?

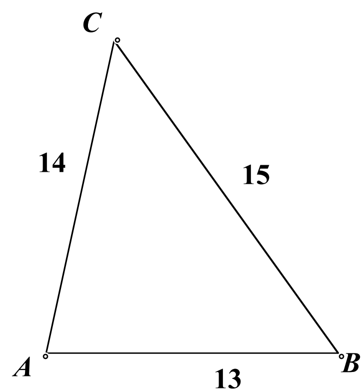
- How can you generate “nice” problems:



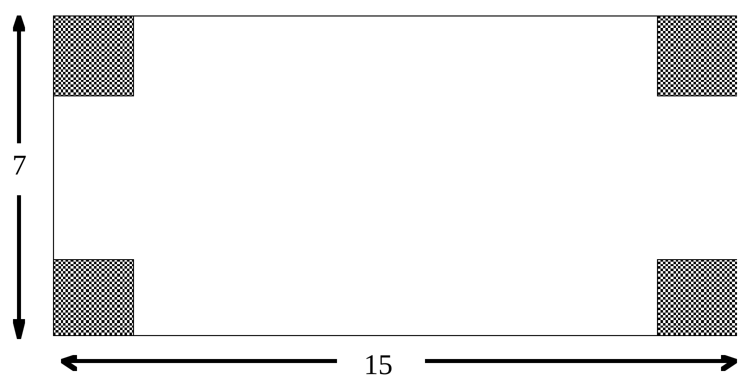
∗ **Nice “law of cosines” problems?** How big is $\angle Q$



∗ **Nice graphing problems?** $f(x) = 140 - 144\,x + 3\,x^2 + x^3$



∗ **Nice area problems?** A $(13, 14, 15)$ triangle



∗ **Nice "box" problems?** Fold up to make a box

- **(How) Can you generate the graph of $y = \sin x$ in Geometer's Sketchpad—without using animation?**

- **(How) Can you use a CAS to get a formula for**
  - $\cos 5x$?
  - $\sum_{k=0}^{n-1} k^5$?
  - The distribution of sums when 4 dice are thrown?

(2) Look at the knowledge used by expert teachers

They know mathematics as a scholar: They have a solid grounding in classical mathematics, including

- its major results
- its history of ideas
- its connections to precollege mathematics

They know mathematics as an educator: They understand the habits of mind that underlie major branches of mathematics and how they develop in learners, including

- algebra and arithmetic
- geometry
- analysis

They know mathematics as a mathematician: They have *experienced the doing of mathematics*—they know what it's like to

- grapple with problems
- build abstractions
- develop theories
- become completely absorbed in mathematical activity for a sustained period of time

They know mathematics as a teacher: Be expert in uses of mathematics that are specific to the profession, including

- the ability "to think deeply about simple things" (*Arnold Ross*)
- the craft of task design
- the ability to see underlying themes and connections
- the "mining" of student ideas

Mathematics for Teaching Demands "Knowing":

- ***The "facts"***
- ***The "epistemology"***
- ***The "experience"*** (∗)
- ***The "craft"*** (∗)

(∗) *are essential pieces missing in many preservice and professional development programs*

(3) Design programs that develop this expertise

(a) The Immersion Experience

PROMYS for Teachers

The Role of Immersion in Professional Development

What is an immersion experience?

Why immersion?

What models can we envision?

Distinguishing features of immersion

- Learning through ***experience***
- Engagement on a deeply ***personal*** level
- Participants create their own ***meaning***
  - Making connections, developing conceptual frameworks
- Requires total ***focus***
- ***Struggling*** with new ideas
- ***Risk-taking*** -- preconceptions lead to deeper understanding
- ***Realistic*** in that there are always ***more questions than answers*** -- an acquired taste.

Program in Mathematics for Young Scientists (PROMYS)

- **Designed as a program for exceptionally talented high school mathematics students.**

- **Founded in 1989 by alumni of the Ross Program**

- **Modeled on the Ross Program, which was originally designed as a program for high school mathematics teachers.**

- **National Recruitment**
  - 70 high school students
  - 18 undergraduate counselors

Program in Mathematics for Young Scientists (PROMYS)

- **Immersion experience of mathematical exploration**
  - six week summer component
  - problem solving in number theory
  - advanced seminars for returning students
  - research projects mentored by mathematicians

- **One student who participated for many summers described participation in the program as "stepping into a mathematical wonderland"**

PROMYS for Teachers

- Began small (1991) with a few pre-service teachers.
- 34 teachers (8 pre-service) in 2004.
- Key collaborations with EDC and PCMI
- PfT participants have designed and delivered the mathematical content section of PCMI's HSTP in each of the last 4 summers.

PROMYS for Teachers

- **Immersion experience of mathematical exploration**
  - Same problem sets, same exams as the students
  - Separate space for teachers to work together
  - Graduate students support teachers
- **Reflection on classroom practice**
  - 5 daylong workshops in academic year at EDC
- **More immersion in mathematics in second summer**

Masters Degree in Mathematics for Teaching

- **Know mathematics as a scholar:**
  - A solid grounding in classical mathematics, including its major results, its history of ideas, its connections to pre-college mathematics
- **Know mathematics as an educator:**
  - Understand the habits of mind that underlie major branches of mathematics and how they develop in learners
- **Know mathematics as a mathematician:**
  - grappling with problems, building abstractions, developing theories
- **Know mathematics as a teacher:**
  - the ability "to think deeply about simple things" (*Arnold Ross*)
  - the creation of classroom activities that uncover central ideas
  - the ability to see underlying themes and connections
  - the "mining" of student ideas

“A lot of us didn't feel we were prepared for the summer program

. . .

Afterwards we felt we could do anything.”

Focus on Mathematics Middle School Teacher, at the end of PROMYS 2004

Why Immersion in Mathematics?

- **The central role of experience**
  - empirical basis of mathematical knowledge
  - personal experience as source of meaning
- **Necessity for open communication -- community**
- **Acquiring taste for hard problems**
- **Learning good judgment** in recognizing significant ideas
  - Knowing what matters
  - Mining of student ideas
- **Sharing ideas with others** in writing and in seminars
- **Questioning answers**

Habits of Thought

- **Acquiring experience**
  - numerical experimentation
  - alert observation
- **Good use of language**
  - asking good questions
  - formulating conjectures
  - proofs and disproofs
- **Review**
  - identifying important ideas
  - Formalization
  - looking for connections
- **Generalization**
  - broadening applicability
  - questioning answers

Paradigms of Learning

How People Learn (NRC study, 1999-2000)

1. Engaging prior understandings

2. Understanding factual knowledge through conceptual frameworks

3. Self-monitoring -- meta-cognition

Mathematics as an Empirical Science

"And it seems to me that the numerous and surprising analogies and that apparently pre-established harmony which the mathematician so often perceives in the questions, methods, and ideas of the various branches of his science, have their origin in this ever-recurring interplay between ***thought and experience.***" -- David Hilbert

Roles for Mathematicians

- **Content experts and instructors**
- **Problem-solving experts**
- **Research mentors**
- **Role models**
  - Mathematics as Living Science
  - Community: contacts with students often last beyond the duration of the program

This is lively, engaging, and satisfying work for mathematicians. Easy to attract mathematicians.

This is valuable work *for* mathematicians (especially young mathematicians)

- Engages them ***as mathematicians***
- Broadens and adds perspective to the ***process of research***
- Provides new ways of thinking about mathematics education

What effects might this have on undergraduate education?

What lessons are to be learned?

- **What is it in the structure of PROMYS that makes it possible to "succeed" with such disparate audiences**?
  - the genius of Arnold Ross's problem sets;
  - the depth of the traditions and the community.
- **Are PROMYS teachers "special" before they begin the program?** Undoubtedly, "yes"!
  - What is special about them?
  - How rare is this brand of "specialness"?
- **What relationship does this have with leadership?**
- **How does the PROMYS experience affect teachers' work in the classroom?**
- **Can we replicate (generalize) key elements of the program?**

Final Remarks

- **The number of “special” mathematics teachers having both significant talent and significant interest in mathematics is significantly higher than is commonly believed.**

- **Helping these teachers is work that mathematicians are uniquely prepared to do.**

- **The mathematical habits of thought required for excellence in teaching are similar to those required for excellence in research.**

- **Mathematicians can benefit *AS MATHEMATICIANS* from engagement in issues of mathematics education.**

(b) The Craft

Thinking deeply about simple things: An example from Lawrence High School

$$
\begin{array}{l}
\quad .142857\ldots \\
7\overline{)\,1.00\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad} \\
\quad\ \underline{7\qquad} \\
\quad\ \mathbf{3}0 \\
\quad\ \underline{28\qquad} \\
\quad\quad \mathbf{2}0 \\
\quad\quad \underline{14\qquad\quad} \\
\quad\quad\ \ \mathbf{6}0 \\
\quad\quad\ \ \underline{56\qquad\ } \\
\quad\quad\quad \mathbf{4}0 \\
\quad\quad\quad \underline{35\qquad} \\
\quad\quad\quad\ \ \mathbf{5}0 \\
\quad\quad\quad\ \ \underline{49\qquad} \\
\quad\quad\quad\quad \mathbf{1}\ \text{here we go again} \ldots
\end{array}
$$

$$\frac{1}{7} = .142857142857\ldots$$

$$\frac{2}{7} = .2857\underline{142857}14\ldots$$

$$\frac{3}{7} = .42857\underline{142857}1\ldots$$

$$\frac{4}{7} = .57\underline{142857}1428\ldots$$

$$\frac{5}{7} = .7\underline{142857}14285\ldots$$

$$\frac{6}{7} = .857\underline{142857}142\ldots$$

$$\frac{7}{7} = .999999999999\ldots$$

.076923 . . .

13) 1.000000000

0

100

91

90

78

120

117

30

26

40

39

1 here we go again . . .

.153846 . . .

13) 2.00000000000

13

70

65

50

39

110

104

60

52

80

78

2 here we go again . . .

$$\frac{1}{13} = .\boxed{076923}\,076923\ldots \qquad \frac{2}{13} = .\boxed{153846}\,153846\ldots$$

$$\frac{10}{13} = .\boxed{769230}\,769230\ldots \qquad \frac{7}{13} = .\boxed{538461}\,538461\ldots$$

$$\frac{9}{13} = .\boxed{692307}\,692307\ldots \qquad \frac{5}{13} = .\boxed{384615}\,384615\ldots$$

$$\frac{12}{13} = .\boxed{923076}\,923076\ldots \qquad \frac{11}{13} = .\boxed{846153}\,846153\ldots$$

$$\frac{3}{13} = .\boxed{230769}\,230769\ldots \qquad \frac{6}{13} = .\boxed{461538}\,461538\ldots$$

$$\frac{4}{13} = .\boxed{307692}\,307692\ldots \qquad \frac{8}{13} = .\boxed{615384}\,615384\ldots$$

Conjecture 1. *When calculating $\frac{1}{n}$ (n not divisible by 2 or 5), the number of fractions in each cyclic pattern is the same, and*

$$\text{the number of cyclic patterns} \times \text{the number fractions in each pattern} = n - 1$$

Question 1 (Joe's Question). *But how do we know that, if n is not divisible by 2 or 5, the first remainder we see again will be 1?*

Question 2 (new and improved Joe's question). *How do we know that, if n is not divisible by 2 or 5, some power of 10 will leave a remainder of 1 when divided by 10?*

Curiosity 1.

$$\frac{1}{19} = .052631578947368421\ldots$$

and

$$\begin{array}{r} 052631578 \\ +\ \ 947368421 \\ \hline 999999999 \end{array} \qquad \begin{array}{r} 052631 \\ 578947 \\ +\ \ 368421 \\ \hline 999999 \end{array} \qquad \begin{array}{r} 052 \\ 631 \\ 578 \\ 947 \\ +\quad 368 \\ 421 \\ \hline 2997 \end{array}$$

Question 3. *Suppose the period for the decimal expansion for* $\frac{1}{n}$ *is* m *and suppose* d *is a divisor of* m. *When will the sum of the digits in the period for* $\frac{1}{n}$, *grouped in sets of* d, *add to a string of* d *9s?*

Conjecture 2 (Carol's conjecture). *If you sum the digits in one period of the decimal expansion for* $\frac{1}{n}$ *(at least when* n *and 10 have no common factors), and keep doing that until you get to a single digit, that digit will always be 9.*

$$\begin{array}{l} \quad\;\; .047619 \\ 21\overline{)\;1.0000000} \\ \quad\;\; \underline{0\quad\;\;} \\ \quad\;\; \mathbf{10}0 \\ \quad\;\; \underline{84\quad} \\ \quad\;\;\; \mathbf{16}0 \\ \quad\;\;\; \underline{147\qquad\quad} \\ \quad\;\;\;\; \mathbf{13}0 \\ \quad\;\;\;\; \underline{126\qquad} \\ \quad\;\;\;\;\; \mathbf{4}0 \\ \quad\;\;\;\;\; \underline{21\qquad} \\ \quad\;\;\;\;\; \mathbf{19}0 \\ \quad\;\;\;\;\; \underline{189\qquad} \\ \quad\;\;\;\;\;\;\; \mathbf{1} \end{array}$$

$$\begin{array}{l} \quad\;\; .095238 \\ 21\overline{)\;2.0000000} \\ \quad\;\; \underline{0\quad\;\;} \\ \quad\;\; \mathbf{20}0 \\ \quad\;\; \underline{189\quad} \\ \quad\;\;\; \mathbf{11}0 \\ \quad\;\;\; \underline{105\qquad\quad} \\ \quad\;\;\;\;\; \mathbf{5}0 \\ \quad\;\;\;\;\; \underline{42\qquad} \\ \quad\;\;\;\;\; \mathbf{8}0 \\ \quad\;\;\;\;\; \underline{63\qquad} \\ \quad\;\;\;\;\; \mathbf{17}0 \\ \quad\;\;\;\;\; \underline{168\qquad} \\ \quad\;\;\;\;\;\;\; \mathbf{2} \end{array}$$

The period for $\frac{1}{21}$ has length *six*. What does 6 have to do with 21? And we have cyclic patterns as before:

$$\frac{1}{21} = .\boxed{047619}\,047619\ldots \qquad \frac{2}{21} = .\boxed{095238}\,095238\ldots$$

$$\frac{10}{21} = .\boxed{476190}\,476190\ldots \qquad \frac{20}{21} = .\boxed{952380}\,952380\ldots$$

$$\frac{16}{21} = .\boxed{761904}\,761904\ldots \qquad \frac{11}{21} = .\boxed{523809}\,523809\ldots$$

$$\frac{13}{21} = .\boxed{619047}\,619047\ldots \qquad \frac{5}{21} = .\boxed{238095}\,238095\ldots$$

$$\frac{4}{21} = .\boxed{190476}\,190476\ldots \qquad \frac{8}{21} = .\boxed{380952}\,380952\ldots$$

$$\frac{19}{21} = .\boxed{904761}\,904761\ldots \qquad \frac{17}{13} = .\boxed{615384}\,615384\ldots$$

Where are the rest of the fractions? We're missing

$$\frac{3}{21}, \frac{6}{21}, \frac{7}{21}, \frac{9}{21}, \frac{12}{21}, \frac{15}{21}, \frac{18}{21}, \text{ and } \frac{14}{21}$$

Of course—these all reduce and will have periods that come from denominator 3 or 7.

Conjecture 3 (new and improved Conjecture 1). *When calculating* $\frac{1}{n}$ *(n not divisible by 2 or 5), the number of fractions in each cyclic pattern is the same, and*

$$\begin{aligned} &\textit{number of cyclic patterns} \times \textit{the number fractions in each pattern} \\ &= \textit{the number of fractions } \frac{k}{n} \ \ (1 \le k < n) \textit{ that don't reduce} \end{aligned}$$

Definition 1.

$$\begin{aligned} \phi(n) = &\textit{ the number of integers between 1 and } n-1 \\ &\textit{ that are relatively prime to } n \end{aligned}$$

Conjecture 4 (newer and improved Conjecture 1). *When calculating $\frac{1}{n}$ (n not divisible by 2 or 5), the number of fractions in each cyclic pattern is the same, and*

$$\textit{the number of cyclic patterns} \times \textit{the number of fractions in each pattern} = \phi(n)$$

APPENDIX 1:

Sample Number Theory Sets

Our problem sets are based on those designed by the late Professor Arnold E. Ross over a 44 year period for use in his ongoing program at the Ohio State University.

"To think deeply of simple things" (A. E. Ross)

PROMYS Number Theory **Problem Set #0** *Boston University, June 18, 2004*

Numerical Problems (*Some food for thought*)

Mathematics, like all of the sciences, is based on experience. This summer we will explore a mathematical system which we have all experienced, namely the system of integers. The integers are the whole numbers

$$\ldots -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ \ldots.$$

We denote the set of integers by the symbol $\mathbf{Z}$. So $\mathbf{Z} = \{\ldots -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ \ldots\}$. As we all know, it is possible to add and multiply in $\mathbf{Z}$. It is also possible to subtract in $\mathbf{Z}$. But it is not always possible to divide evenly in $\mathbf{Z}$. For example, if we try to divide 1 by 2 in $\mathbf{Z}$, then we seek an integer $x \in \mathbf{Z}$ such that $2x = 1$. But there is no such integer x. Early in school, most of us learn how to do long division with remainders. This is a simple process known to mathematicians as the *division algorithm*. For example, if we divide 1068 by 7 then we get a quotient of 152 and a remainder of 4.

$$\begin{array}{r|l} & \ \ 152 \quad = \textit{quotient} \\ \cline{2-2} 7 & 1068 \\ & 1064 \\ \cline{2-2} & \quad\ 4 \quad = \textit{remainder} \end{array} \qquad\qquad 1068 = 7 \cdot 152 + 4$$

More generally, if m is a fixed positive integer, then we can divide any integer a by m to get a quotient q and a remainder r where r is one of the numbers $0,\ 1,\ 2, \ldots,\ m-1$. We summarize this by writing $a = m \cdot q + r$ where $0 \leq r < m$.

The division algorithm is a simple process. Nevertheless, as we shall see this summer, it has a large number of unexpectedly deep consequences and can be generalized in far-reaching ways. One of the most basic applications of the division algorithm is to our notation for integers. It is customary to use base ten notation to write down the elements of $\mathbf{Z}$. For example 1068 is our notation for $1 \cdot 10^3 + 0 \cdot 10^2 + 6 \cdot 10 + 8$. But we could equally well use the base seven or some other base. As an example, let's write 1068 to base 7. We begin by using the division algorithm to divide 1068 by 7. We have already done this above, where we obtained $1068 = 7 \cdot 152 + 4$. Next we divide the quotient 152 by 7 to get a new quotient 21 and a new remainder 5: $152 = 7 \cdot 21 + 5$. Continuing in this way, we obtain the following list of equations.

$$\begin{array}{ccccc} 1068 & = & 7 \cdot 152 & + & 4 \\ 152 & = & 7 \cdot 21 & + & 5 \\ 21 & = & 7 \cdot 3 & + & 0 \\ 3 & = & 7 \cdot 0 & + & 3. \end{array}$$

From these calculations we deduce that $1068 = 3 \cdot 7^3 + 0 \cdot 7^2 + 5 \cdot 7 + 4$. Hence 1068 (to base ten) is equal to $(3054)_7$ written to base 7. The four 'digits' in this base seven representation are the four remainders above.

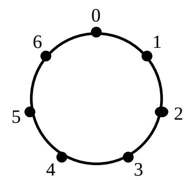
P1: Let $N = 32701$ to base ten. Write N to base two, to base three, to base eleven. In the last case, introduce the new "digit", $T = 10$ if necessary.

P2: Without change of base, (a) add $(6153)_7$ to $(3455)_7$; (b) subtract $(2346)_7$ from $(4354)_7$; (c) multiply $(632)_7$ by $(435)_7$; (d) divide $(5602)_7$ by $(43)_7$. Here the base is seven throughout.

Exploration

To help us gain insight and perspective on $\mathbf{Z}$, we will also investigate a number of alternative systems of arithmetic. Some fundamental examples of systems of arithmetic are given by so-called *modular arithmetics*. Probably the easiest way to explain what we mean by a modular arithmetic is to first look at a simple example.

Consider the set $\mathbf{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$. This is the set of possible remainders we obtain if we divide by 7 using the division algorithm. The number 7 is called the modulus of our system, and $\mathbf{Z}_7$ is called the system of least non-negative residues modulo 7. We are going to equip $\mathbf{Z}_7$ with two operations: addition and multiplication. To describe these operations, it is convenient to arrange the elements of $\mathbf{Z}_7$ in a circle, as follows.



Now suppose we want to add two elements of $\mathbf{Z}_7$, say we want to add 5 to 6. Then we start at 6 on the above circle and proceed clockwise a total of 5 units. That brings us to 4. So we say $5 + 6 = 4$ in $\mathbf{Z}_7$. Here are some more examples of addition in $\mathbf{Z}_7$.

$$1 + 2 = 3; \quad 6 + 6 = 5; \quad 2 + 5 = 0; \quad 3 + 4 + 5 + 6 = 4.$$

It is also possible to multiply in $\mathbf{Z}_7$. Suppose for example, we want to multiply 6 by 2. Then we just add 6 to itself twice: $6 \cdot 2 = 6 + 6 = 5$. Similarly, to multiply 5 by 4 we add 5 to itself 4 times: $5 \cdot 4 = 5 + 5 + 5 + 5 = 6$. Here are a few more examples of multiplication in $\mathbf{Z}_7$.

$$2 \cdot 3 = 6; \quad 3 \cdot 4 = 5; \quad 4 \cdot 6 = 3; \quad 5 \cdot 5 \cdot 5 = 6.$$

Once we understand addition and multiplication, we can try to do more interesting calculations in $\mathbf{Z}_7$.

P3: How many of the following can you find in $\mathbf{Z}_7$?

$$4 \cdot 5,\ 2 - 6,\ 1/2,\ 2/5,\ \sqrt{2},\ \sqrt{-3},\ \sqrt{-1},\ \sqrt[3]{6}\ ?$$

To get you started on this one, let's take a look at the third example: What is $1/2$ in $\mathbf{Z}_7$? To answer this question, we must first realize that $1/2$ stands for the multiplicative inverse of 2 in $\mathbf{Z}_7$. Thus $1/2$ is an element x in $\mathbf{Z}_7$ for which $2x = 1$. Can you find an element $x \in \mathbf{Z}_7$ such that $2x = 1$? Is there more than one such x? Note that $\mathbf{Z}_7$ has has only seven elements. If all else fails, you can always just try out all seven possible values of x.

P4: How many of the following can you find in $\mathbf{Z}_{11}$? In $\mathbf{Z}_{35}$? In $\mathbf{Z}_9$? Any conjectures?

$$7 + 8,\ 4 - 9,\ 3 \cdot 5,\ 5^2,\ 5^3,\ 1/5,\ 3/8,\ \sqrt{3},\ \sqrt{-2},\ \sqrt{-6}.$$

Ingenuity

P5: Can you find an integer $n > 1$ such that the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

is an integer?

[I]t is impossible to be a mathematician without being a poet in soul ... [T]he poet has only to perceive that which others do not perceive, to look deeper than others look. And the mathematician must do the same thing.
– Sonya Kovalevskaya

PROMYS Number Theory **Problem Set #1** *Boston University, July 5, 2004*

Reading Search

Q1: What is a perfect number? Give four examples of perfect numbers.

Q2: What is a Mersenne prime? Give four examples of Mersenne primes.

Q3: What is the meaning of $\sum_{\substack{d|N \\ d>0}} \frac{1}{d}$? Compute the value of this sum for some small values of N.

Asking Good Questions

P1: Compare the following mathematical systems with each other: $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, $2\mathbf{Z}$, $\mathbf{Z}_3$, $\mathbf{Z}_6$, $\mathbf{Z}_8$, $\mathbf{Z}_{11}$. Each of these systems has two operations (addition and multiplication). Which of these systems resemble each other in regard to the essential properties of the operations?

Exploration

P2: Consider the set of all polynomials in x with coefficients in $\mathbf{Z}_3$. Denote this set of polynomials by $\mathbf{Z}_3[x]$. Consider $f(x) = x^2 + 2x + 1 \in \mathbf{Z}_3[x]$ and $g(x) = 2x^2 + x + 1 \in \mathbf{Z}_3[x]$. Calculate $f(x) + g(x)$ and $f(x) \cdot g(x)$. Factor $x^4 - 1$ into linear factors in $\mathbf{Z}_5[x]$ and $x^6 - 1$ into linear factors in $\mathbf{Z}_7[x]$.

Prove or Disprove and Salvage if Possible

In resolving questions and problems posed in our problem sets, one has to make use of many properties which one associates with integers. Make a list (inventory) of such properties and see if this inventory suffices for every discussion of questions of arithmetic which we undertake. If this inventory proves to be adequate for such discussion, then we shall consider it as an acceptible description of the integers.

P3: $a|a$ for every $a \in \mathbf{Z}$.

P4: $a|b \Rightarrow b|a$ for all a and b in $\mathbf{Z}$.

P5: For all $a, b, c \in \mathbf{Z}$, if $a|b$ and $b|c$ then $a|c$.

P6: $a|b \Rightarrow a|bc$ for all integers a, b, c.

P7: For all a, b, c in $\mathbf{Z}$, if $a|b$ and $a|c$ then $a|(b + c)$.

P8: For all $a, b, c \in \mathbf{Z}$, $a|bc \Rightarrow a|b$ or $a|c$.

Numerical Problems (*Some food for thought*)

P9: Using division to base seven, write $N = (34652)_7$ to base two, to base five, to base eleven.

P10: Write each of the following numbers to base three: 3, 9, 27, 243, 1/3, 1/9, 1/27. Write each of these numbers to base two. Any conjectures?

P11: Find the following elements in $\mathbf{Z}_5$: $-1,\ 1/2,\ 2/3,\ \sqrt{-1}$. How many of these elements can you find in $\mathbf{Z}_6$? in $\mathbf{Z}_{10}$? in $\mathbf{Z}_{11}$? in $\mathbf{Z}_{13}$?

P12: Make a list of the perfect squares in $\mathbf{Z}_5$, in $\mathbf{Z}_{17}$, in $\mathbf{Z}_{19}$, in $\mathbf{Z}_{21}$. How many squares are there in each case? Any conjectures?

P13: Calculate the sums: $\sum_{\substack{d|6 \\ d>0}} \frac{1}{d}$, $\sum_{\substack{d|28 \\ d>0}} \frac{1}{d}$. Can you make an interesting conjecture? Perhaps another example may help: $\sum_{\substack{d|496 \\ d>0}} \frac{1}{d}$. Calculate this sum and compare its value to the values of the first two sums.

The Art of Counting

P14: Consider the set $S = \{a, b, c, d\}$. How many distinct subsets of S are there? How many distinct subsets of the set $T = \{c, d, e, f, g, h\}$ are there? The sets S and T have 4 and 6 elements respectively. How many elements are there in $S \cap T$? in $S \cup T$? In general, if you know the number of elements in each one of two given sets S and T, can you say how many elements there are in $S \cup T$?

APPENDIX 2 :

What is Knowledge of Mathematics for Teaching?

Does the Focus on Mathematics Immersion Program achieve it? How can we assess it? Early Evidence

Sabra Lee and Carol Baldassari
Program Evaluation and Research Group
Lesley University

As the external evaluators on the Focus on Mathematics [FoM] Math/Science Partnership, we have been documenting the development of the partnership, the implementation of their professional development programs, and collecting data from participating middle and high school mathematics teachers from Arlington, Chelsea, Lawrence, Waltham and Watertown school districts.

Last fall, we conducted a set of in-depth interviews with seven ‘mathematics teaching fellows’ who participated in last summer’s immersion program, called PROMYS for Teachers, and who have enrolled in the new masters program at Boston University. The teachers were diverse in terms of their educational background [the extent to which they had studied mathematics in undergraduate or graduate programs], the number of years they have taught mathematics, and the grade levels they currently teach. Interviewees included both middle and high school mathematics teachers. All five of the districts that are partners in the Focus on Mathematics Program were represented.

Our preliminary evidence documents that teachers in the Focus on Mathematics immersion program are deepening their knowledge of mathematics for teaching and are beginning to identify new instructional approaches based on that learning, and, in some cases, they are integrating those into their classroom practices.

Teachers’ comments on their experience in the summer program, what they gained from it, and their reports on how that experience was influencing their approach to teaching mathematics can be found following the discussion of our next steps, below.

Next Steps:
To study teachers’ knowledge of mathematics for teaching and how it is manifested, we are designing case studies of 6 teachers engaged in Focus on Mathematics immersion program to look how teachers involved in the program:

- Deepen their knowledge of mathematics for teaching
- Shift their approach to teaching mathematics
- Provide opportunities for students to work on mathematical problems and projects of real substance
- Influence mathematics teaching and learning in their schools and districts

With extended support we hope to

- Develop processes for observing and documenting changes in their work with students
- Develop and pilot classroom observation protocols as well as performance assessments to assess knowledge of mathematics for teaching
- Examine their efforts to influence and lead district change, and the impact of those efforts.

We welcome comments and suggestions from and collaborations with members of the MSP community.

Focus on Mathematics Teachers' Comments on the PROMYS for Teachers Summer Institute, 2004.

> One of the things the mathematician threw at us was fundamental calculus. I am seeing more connections. Calculus is about rate of change. In one sense, kids at the elementary level are doing calculus, they're just not calling it that. The mathematician we are working with is showing us that all these different subjects within mathematics are tied together much more than I thought.
>
> It [summer immersion program] was learning that discovery feeling -- with your own ability, coming up with your own conclusions. It also gave you a deep understanding of the things we teach -- the everyday things. [It was an opportunity] to delve in and understand the theory behind them, for our teaching.
>
> I was the student now, who had totally foreign material in front of me that I needed to learn - difficult concepts. Some I couldn't get. It made me understand how my students feel when I'm teaching something and they are not grasping it. It put me in their shoes and it made me think about how I ask questions and how I learn. It made me think about how I can help my students learn to be more independent, and how to attack things that they have no idea about, where to start. It gave me a way to help them approach things they don't know.
>
> They'd ask us a series of questions and it was up to us to take those questions and see what happened. At first it was frustrating, then we enjoyed it. No one explicitly tells us the answers. It was constructive, it encouraged us to dig deeper. I wouldn't have wanted too much support; it would have cheated the experience. I don't know that that would have helped us grow any further.
>
> I became very sensitive to the emotional aspects of learning and math. Most math teachers love math but over the summer I was sometimes close to tears. My students also get frustrated to the point of tears. I think it is a great sign because they are feeling the emotional aspect and I think if we can turn that frustration into a level of excitement. I think PROMYS was showing clues about how to redirect those frustration points or get our students to work past those points. This summer gave me new ideas. If you want a powerful math community you have to find a way to get students hooked or have some feeling toward it and see the beauty in it. With number theory you start to see the beauty.
>
> The first weeks of the program, I could connect to things I knew. Even if I was frustrated one day, the next day I'd have an epiphany. There were lots of ups and downs. Understanding math concepts was not enough. You had to look at things in different ways. It's not necessarily intuitive. I learned a lot about my own patience. Every time I felt frustrated I realized something I would not have realized without being frustrated.

I learned a lot about number theory and fell in love with it … and computation and understanding of numbers, and counting. My understanding of number sense is much stronger and it is not just out of memorizing things but seeing what happens and why with numbers.

One powerful lesson I learned will help me understand how to diagnose students to determine what they know. I realized that with anything abstract given to me I couldn't work with algebraic equations but had to do numericals first. I had to prove it was true in the first place with number. Then algebraic representation came to me. Teaching algebra to grade 8 and kids that are all over place with their math development, this will really help me with those not ready for more abstract stuff. I have to give them more numerical work until they say, "I know what is going to happen."

Part of it was self-discovery. Like for me, how to attack something I have no idea about. That really helped.

A lot of us didn't feel we were prepared for the summer program. Afterwards felt we could do anything.

I'm looking forward to doing it again next summer. I like the idea of going back next summer and doing the problem sets again and work on all the things I didn't get to work on this summer. Sometimes, the connections I'd seen lasted only for a split second.

I've taken some of these proofs, some of the more accessible proofs, and incorporated them into my teaching. I tell them and myself don't accept things as established facts, wait until you see why it works.

There's no class like it. I was exposed to the new way of letting someone go, providing them with all the questions and none of the answers. At PfT, they are guiding the learning but not explicitly stating it. It showed me I can learn in a different way.

I will find places to build number sense. The first week I gave them blank multiplication tables. I asked them to do them as homework – they said no problem. Next day I said how many did not need to do out all the multiplication because they saw patterns? Hands went up. I told them, for each problem you can explain what is happening, I'll give a point on the next test. I will integrate more things like that and they love it.

I believe it already has [changed my teaching], I've been teaching more number theory embedded in Algebra. I used a whole section on place value and powers of 10 and kids liked seeing the deeper value of how division works or multiplication works. This is only my 2nd year teaching but I will continue to develop as I go through the program -- to explore more deeply certain things. I've already started this year, It's exciting to already use this stuff and every year I could use more. Going back [to the program] next summer will help with that even more.

Some of the things I learned when I was in high school never made sense to me, but after this summer, I have an understanding of prime numbers and composite numbers that I didn’t really understand before.

For more information, visit our website: www.focusonmath.org

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