Investigating
The Simplex Lock:
A Mathematical Adventure

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The Simplex Lock

- 5-Button mechanical device
- Advertising claims “Thousands of combinations”
- How many are there?
- Let’s investigate!
Simplex Lock Combinations

- A combination is a sequence of 0–5 pushes using 0-5 buttons
- Each button is used at most once (once pushed, it stays in)
- Each push may include any of the buttons that have not yet been pushed, up to/including all remaining buttons
- A combination need not include all buttons ("unlocked" = 0 buttons, 0 pushes)
- When 2 or more buttons are pushed simultaneously, order does not matter
Combination Terminology

- “|” delineates buttons pushed in sequence
  - “1|2|3” uses 3 buttons, in order, 3 pushes
  - “3|2|1” is a different combination

- “+” groups buttons pushed simultaneously
  - “1+2+3” uses 3 buttons simultaneously, 1 push
  - Order not a factor: “1+2+3” = “3+2+1”

- “1|2+3” uses 3 buttons, 2 pushes
  - Push 1, then 2+3 simultaneously
  - “1|2+3” = “1|3+2”
  - “2+3|1” is a different combination
Examples of Simplex Lock Combinations

- “1+5|2|3|4” (5 buttons, 4 pushes)
- “2|1+5|3|4” (5 buttons, 4 pushes)
- “1+2+3+4+5” (5 buttons, 1 push)
- “1|2|3|4|5” (5 buttons, 5 pushes)
- “2|1|3|4|5” (5 buttons, 5 pushes)
- “4+2” (2 buttons, 1 push)
- “4+5” (2 buttons, 1 push)
- “1+3|4+5” (4 buttons, 2 pushes)
## Comparing Mechanical Locks

### Comparison of Locks and Their Numbers of Combinations

<table>
<thead>
<tr>
<th>Lock Type</th>
<th>Number of Combinations</th>
</tr>
</thead>
<tbody>
<tr>
<td>5-Button Simplex Lock</td>
<td>000, 64 40</td>
</tr>
<tr>
<td>Master Combination Lock</td>
<td>$40^3 \approx 64,000^*$</td>
</tr>
<tr>
<td>Barrel Combination Lock</td>
<td>$10^4 = 10,000$</td>
</tr>
</tbody>
</table>
Problem Solving Strategies

- Simplify
  - Start with a simpler problem

- Organize
  - Tables
  - Enumeration strategies; count by:
    - Buttons used (0-5)
    - Pushes used (0-5)
    - Shape (xxx, x|xx, xx|x, x|x|x)

- Generalize
Strategy: Simplify

☐ List the combinations on a 1-button Simplex Lock
  ■ Unlocked (0 buttons, 0 pushes)
  ■ 1 (1 button, 1 push)

☐ Now, list the combinations on a 2-button Simplex Lock
Strategy: Simplify & Organize

- Combinations on a 2-button Simplex Lock
  - Unlocked (0 buttons, 0 pushes)
  - 1 (1 button, 1 push)
  - 2 (1 button, 1 push)
  - 1+2 (2 buttons, 1 push)
  - 1|2 (2 buttons, 2 pushes)
  - 2|1 (2 buttons, 2 pushes)

- Now, list the combinations on a 3-button Simplex Lock
Factorial Notation

- $n!$ is the product of all positive integers less than or equal to $n$
- $3! = 3 \times 2 \times 1$
  - A set of 3 elements can be arranged 6 ways
- $1! = 1$
- Recall that $0! = 1$
  - For students who are confused by this, I remind them that there is still 1 way to arrange an empty set of elements (see 0-button lock)
Combinatorics: Permutations

- Permutation: ordered *arrangement* of elements from a set
- In how many ways can a set of 3 elements be *arranged*?
  - List them
  - Any ways to count these?
Combinatorics: Permutations

- $P(n,r)$ is the number of possible arrangements of $r$ elements from a set of $n$ elements

\[ P(n,r) = \frac{n!}{(n-r)!} \]

- On a 3-button lock, the number of sequential 2-button arrangements (example: “1|2”) is $P(3,2)$

\[ P(3,2) = \frac{3!}{(3-2)!} = 6 \]

- On a 3-button lock, the number of sequential 3-button arrangements (example: “1|2|3”) is $P(3,3)$

\[ P(3,3) = \frac{3!}{(3-3)!} = 6 \]
Combinatorics: Combinations

- “\( n \) choose \( k \)” is the number of ways to choose \( k \) elements (combinations) from a set of \( n \) elements

\[
\binom{n}{k} = \frac{n!}{(n-k)!k!}
\]

- On a 5-button lock, the number of 3-button, 1-push combinations (example: “2+5+3”) is “5 choose 3”

\[
\binom{5}{3} = \frac{5!}{2!3!} = 10
\]
Combinatorial Notation

- On a 5-button lock, “1|2+3” is “5 choose 1, 4 choose 2”
  \[
  \binom{5}{1} \binom{4}{2} = \frac{5!}{4!1!} \frac{4!}{2!} = \frac{5!4!}{4!1!2!} = 30
  \]

- On a 5-button lock, “1+2|3” is “5 choose 2, 3 choose 1”
  \[
  \binom{5}{2} \binom{3}{1} = \frac{5!}{3!2!} \frac{3!}{1!2!} = \frac{5!3!}{3!2!2!1!} = 30
  \]

- Since these behave identically, we simplify counting of permutations such as “1|2+3” and “1+2|3” as follows
  \[
  2 \binom{5}{1} \binom{4}{2} = 2(30) = 60
  \]
Choose Numbers in Pascal’s Triangle

- For “$n$ choose $k$” find row $n$, entry $k$
  - The first row is numbered 0
  - The first entry in a row is numbered 0

- “4 choose 2” = 6
### 3-Button Lock: Organize (by # buttons)

<table>
<thead>
<tr>
<th>Buttons used (k)</th>
<th>Example</th>
<th>Notation</th>
<th>$\Sigma(k)$</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Unlocked</td>
<td>$\binom{3}{0} = 1$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1 or 2 or 3</td>
<td>$\binom{3}{1} = 3$</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>$\binom{3}{1}\binom{2}{1} = 6$</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>1+2</td>
<td>$\binom{3}{2} = 3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>$\binom{3}{1}\binom{2}{1}\binom{1}{1} = 6$</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2+3</td>
<td>$2\binom{3}{1}\binom{2}{2} = 2(3) = 6$</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>1+2</td>
<td>3</td>
<td>$\binom{3}{3} = 1$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1+2+3</td>
<td>$\binom{3}{3} = 1$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### 3-Button Lock: Organize (by # pushes)

<table>
<thead>
<tr>
<th># Pushes</th>
<th>Example</th>
<th>Notation</th>
<th>$\Sigma(pushes)$</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Unlocked</td>
<td>$\binom{3}{0}=1$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1 or 2 or 3</td>
<td>$\binom{3}{1}=3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1+2</td>
<td>$\binom{3}{2}=3$</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1+2+3</td>
<td>$\binom{3}{3}=1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>$\binom{3}{1}\binom{2}{1}=6$</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2+3</td>
<td>$2\binom{3}{1}\binom{2}{2}=2(3)=6$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1+2</td>
<td>3</td>
<td>$\binom{3}{1}\binom{2}{1}\binom{1}{1}=6$</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>$\binom{3}{1}\binom{2}{1}\binom{1}{1}=6$</td>
</tr>
</tbody>
</table>
# The 5-Button Simplex Lock

## 5-button lock using 0 buttons

<table>
<thead>
<tr>
<th>Shapes</th>
<th># Combinations</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1 (unlocked)</td>
</tr>
</tbody>
</table>

## 5-button lock using 1 button

<table>
<thead>
<tr>
<th>Shapes</th>
<th># Combinations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \binom{5}{1} = 5 )</td>
</tr>
</tbody>
</table>
The 5-Button Simplex Lock

<table>
<thead>
<tr>
<th>Shapes</th>
<th># Combinations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1+2</td>
<td>( \binom{5}{2} = 10 )</td>
</tr>
</tbody>
</table>
The 5-Button Simplex Lock

<table>
<thead>
<tr>
<th>Shapes</th>
<th># Combinations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>2+3</td>
</tr>
<tr>
<td>1+2</td>
<td>3</td>
</tr>
<tr>
<td>1+2+3</td>
<td>( \binom{5}{3} = 10 )</td>
</tr>
</tbody>
</table>
## The 5-Button Simplex Lock

### 5-button lock using 4 buttons

<table>
<thead>
<tr>
<th>Shapes</th>
<th># Combinations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>2+3</td>
</tr>
<tr>
<td>1+2</td>
<td>3</td>
</tr>
<tr>
<td>1+2</td>
<td>3+4</td>
</tr>
<tr>
<td>1</td>
<td>2+3+4</td>
</tr>
<tr>
<td>1+2+3</td>
<td>4</td>
</tr>
<tr>
<td>1+2+3</td>
<td>4</td>
</tr>
</tbody>
</table>
# The 5-Button Simplex Lock

<table>
<thead>
<tr>
<th># Buttons Used</th>
<th># Shapes</th>
<th># Combinations</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>30</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>130</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>375</td>
</tr>
<tr>
<td>5</td>
<td>?</td>
<td>???</td>
</tr>
</tbody>
</table>
The 5-Button Simplex Lock

- Find numbers of combinations
- Investigate the patterns
- Make conjectures & generalizations
  - Shapes for \( n \)-button combinations
  - Combinations using \( n \) buttons
  - Combinations using \( k \) pushes
- Prove ‘em
- Are there formulas?
- Excel, MATLAB etc.
- Extensions
In the 1960’s, German-born mathematician Kurt Mahler introduced a form of polynomial expansions of binomial coefficients of the form \( \binom{x}{n} \) where \( n \) are positive integers:

\[
\begin{align*}
\binom{x}{1} &= \frac{x!}{(x-1)!!} = \frac{x}{1} = x \\
\binom{x}{2} &= \frac{x!}{(x-2)!2!} = \frac{(x)(x-1)}{2!} = \frac{x^2-x}{2} \\
\binom{x}{3} &= \frac{x!}{(x-3)!3!} = \frac{(x)(x-1)(x-2)}{3!} = \frac{(x-2)(x^2-x)}{6} = \frac{x^3-3x^2+2x}{6} \\
\binom{x}{4} &= \frac{x!}{(x-4)!4!} = \frac{(x)(x-1)(x-2)(x-3)}{4!} = \frac{(x-3)(x^3-3x^2+2x)}{24} = \frac{x^4-6x^3+11x^2-6x}{24} \\
\binom{x}{5} &= \frac{x!}{(x-5)!5!} = \frac{(x)(x-1)(x-2)(x-3)(x-4)}{5!} = \frac{(x-4)(x^4-6x^3+11x^2-6x)}{120} = \frac{x^5-10x^4+35x^3-50x^2+24x}{120}
\end{align*}
\]
Simplex Lock Extensions

- Writing each of these Mahler expansions in terms of relative powers of $x$ we have:

\[
x^1 = 1 \binom{x}{1}
\]
\[
x^2 = 2 \binom{x}{2} + 1 \binom{x}{1}
\]
\[
x^3 = 6 \binom{x}{3} + 6 \binom{x}{2} + 1 \binom{x}{1}
\]
\[
x^4 = 24 \binom{x}{4} + 36 \binom{x}{3} + 14 \binom{x}{2} + 1 \binom{x}{1}
\]
\[
x^5 = 120 \binom{x}{5} + 240 \binom{x}{4} + 150 \binom{x}{3} + 30 \binom{x}{2} + 1 \binom{x}{1}
\]
Simplex Lock Extensions

Comparing the 3rd Mahler expansion with the combinations of a 3-Button Simplex Lock *that utilize all 3 buttons*, we see some interesting connections:

\[ x^3 = 6 \binom{x}{3} + 6 \binom{x}{2} + 1 \binom{x}{1} \]

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>2+3</td>
</tr>
<tr>
<td></td>
<td>1+2</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>1+2+3</td>
<td>(\binom{3}{3} = 1)</td>
</tr>
</tbody>
</table>
My Own Current Research

- Closed form Simplex Lock formula?
- Comparing Simplex Lock and Mahler basis
- Proving connection between Mahler basis and Simplex Lock combinations
- Investigating “Pascal-like” triangles generated by Simplex Lock combinations and Mahler basis terms
The Simplex Lock Problem

☐ Real world problem
☐ Open-ended investigation
☐ Entry at various skill levels
☐ Cooperative learning & sharing
☐ Multiple levels of exploration at various levels of interest, skill & time
The Simplex Lock Problem

☐ Thanks to Al Cuoco at EDC, and to Brian Harvey at UC Berkeley
☐ Special thanks to Professor Robert Pollack at Boston University
☐ More info on this and other cool projects at EDC website: http://www2.edc.org/makingmath/mathproj.asp

☐ Contact me at adkatz@bu.edu